Convergence of Power Series

Lecture Notes

Consider a power series, say

$$f(x) \;=\; 1 \,+\, x \,+\, x^2 \,+\, x^3 \,+\, x^4 \,+\, \cdots.$$

Does this series converge? This is a question that we have been ignoring, but it is time to face it.

Whether or not this power series converges depends on the value of x. If x is too large, then the series will diverge:

$$f(10) \;=\; 1 \,+\, 10 \,+\, 100 \,+\, 1000 \,+\, \cdots \;=\; \infty.$$

However, if x is small enough, then the series will converge:

$$f(0.1) \;=\; 1 \,+\, 0.1 \,+\, 0.01 \,+\, 0.001 \,+\, \cdots \;=\; 1.111111\ldots$$

In fact, since this particular series is geometric, it will converge whenever $|x| < 1$, and diverge whenever $|x| > 1$.

In general, a power series converges whenever x is close to 0, and may diverge if x is far away from 0. The maximum allowed distance from 0 is called the ***radius of convergence***.

Series With Negative Terms

So far, almost all of our discussion of convergence and divergence has involved positive series. However, since power series often have negative terms, we must begin by discussing the convergence of series whose terms are negative.

Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \;=\; 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \cdots$$

This series is similar to the series $\sum 1/n^2$, except the $(-1)^{n+1}$ causes the terms to alternate between positive and negative. So, do you think this series converges?

Of course it does! Since the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \;=\; 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$$

converges, it cannot possibly become divergent if we change some of the terms to negative. This reasoning leads to the following rule:

ABSOLUTE CONVERGENCE RULE

Let $\sum a_n$ be a series with some negative terms, and consider the positive series obtained by taking the absolute value of each term:

$$\sum |a_n| \;=\; |a_1| + |a_2| + |a_3| + \cdots.$$

If the positive series $\sum |a_n|$ converges, then the series $\sum a_n$ must converge as well.

EXAMPLE 1 Determine whether the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n^3+3n+1}$ converges.

SOLUTION The associate positive series is

$$\sum_{n=0}^{\infty} \left| \frac{(-1)^n}{n^3+3n+1} \right| \;=\; \sum_{n=0}^{\infty} \frac{1}{n^3+3n+1}.$$

Since this positive series converges, the original series must converge as well. ■

Be aware that the absolute convergence rule only works for *convergence*. If the positive series $\sum |a_n|$ diverges, it is still possible for the series $\sum a_n$ to converge. One famous example is the **alternating harmonic series**:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \;=\; 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

Though the harmonic series itself diverges, the alternating harmonic series actually converges! This is a direct result of the subtractions: each term partially cancels with the previous term, resulting in a total sum that is finite. Though you should be aware of this phenomenon, we will not be discussing it in great detail. (If you are interested, you can read more about it near the beginning of section 8.4 in the textbook.)

Mostly we will be using the following test, which combines the absolute convergence rule with the root test:

ROOT TEST (ABSOLUTE VALUE FORM)

Let $\sum a_n$ be a series, and let

$$r \;=\; \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

(a) If $r < 1$, then the series $\sum a_n$ converges.

(b) If $r > 1$, then the series $\sum a_n$ diverges.

(c) If $r = 1$, then the root test is inconclusive.

When $r < 1$, this test works because of the absolute convergence rule. When $r > 1$, this test works because the terms themselves do not approach zero.

EXAMPLE 2 Determine whether the series $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2 3^n}$ converges.

SOLUTION We start by taking the absolute value to eliminate the negatives:

$$\left| \frac{(-2)^n}{n^2 3^n} \right| = \frac{2^n}{n^2 3^n}$$

Using this absolute value,

$$r = \frac{(2)}{(1)(3)} = \frac{2}{3}$$

Since $2/3 < 1$, this series converges. ■

Radius of Convergence

You can use the root test to determine which values of x make a power series converge. It is important to remember that x may be negative, so:

$$\lim_{n\to\infty} \sqrt[n]{|x^n|} = |x|$$

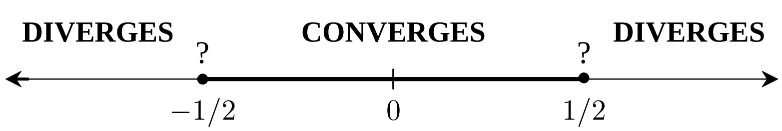
EXAMPLE 3 For what values of x does the following series converge?

$$1 + 2x + 4x^2 + 8x^3 + 16x^4 + \cdots$$

SOLUTION This is the series $\sum_{n=0}^{\infty} 2^n x^n$. Using the root test,

$$r = \lim_{n\to\infty} \sqrt[n]{|2^n x^n|} = 2|x|.$$

The series converges for $r < 1$, which happens when $|x| < \frac{1}{2}$. Similarly, the series diverges when $r > 1$, which happens when $|x| > \frac{1}{2}$. This gives us the following picture:



All we need to determine is what happens for $x = \pm\frac{1}{2}$. (The root test gives $r = 1$ for these two values of x, which is inconclusive.) Here is what the series looks like for these values of x:

$$x = +\frac{1}{2}: \qquad 1 + 1 + 1 + 1 + 1 + \cdots$$

$$x = -\frac{1}{2}: \qquad 1 - 1 + 1 - 1 + 1 - \cdots$$

As you can see, the series diverges for both $x = \frac{1}{2}$ and $x = -\frac{1}{2}$. Therefore, the power series converges only if $-\frac{1}{2} < x < \frac{1}{2}$. ■

For the series above, the root test determines that the series converges for $|x| < \frac{1}{2}$ and diverges for $|x| > \frac{1}{2}$. This is always the sort of information that the root test provides:

RADIUS OF CONVERGENCE

Let $\sum c_n x^n$ be a power series. Then there exists a radius R for which

(a) The series converges for $|x| < R$, and

(b) The series converges for $|x| > R$.

R is called the **radius of convergence.**

Do not confuse the capital R (the radius of convergence) with the lowercase r (from the root test). **They are completely different.** ***R* stands for radius.** In the last example, r turned out to be $2|x|$, which resulted in a radius of $R = \frac{1}{2}$.

EXAMPLE 4 Find the radius of convergence for the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\,3^n} x^n$

SOLUTION Using the root test:

$$r \;=\; \lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{n\,3^n} x^n\right|} \;=\; \frac{(1)}{(1)(3)}\,|x| \;=\; \frac{1}{3}|x|$$

As you can see, the series converges if $|x| < 3$, and diverges if $|x| > 3$. Therefore, the radius of convergence is $R = 3$. ■

EXAMPLE 5 Find the radius of convergence for the series $\sum_{n=0}^{\infty} \frac{5^n}{n^2+1} x^n$

SOLUTION Using the root test:

$$r = \lim_{n\to\infty} \sqrt[n]{\left| \frac{5^n}{n^2+1} x^n \right|} = \frac{(5)}{(1)} |x| = 5|x|$$

As you can see, the series converges if $|x| < \frac{1}{5}$, and diverges if $|x| > \frac{1}{5}$. Therefore, the radius of convergence is $R = \frac{1}{5}$. ■

The following example has infinite radius of convergence.

EXAMPLE 6 Find the radius of convergence for the series $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$

SOLUTION Using the root test:

$$r = \lim_{n\to\infty} \sqrt[n]{\left| \frac{1}{n!} x^n \right|} = \frac{1}{(\infty)} |x| = 0$$

Since $r = 0$ no matter what x is, the series converges for any value of x. Therefore, the radius of convergence is $R = \infty$. ■

EXERCISES

1–12 ■ Find the radius of convergence of the series.

1. $\sum_{n=2}^{\infty} \frac{1}{10^n \ln n} x^n$

2. $\sum_{n=1}^{\infty} \frac{\ln n}{3^n} x^n$

3. $\sum_{n=0}^{\infty} n^2 2^n x^n$

4. $\sum_{n=1}^{\infty} \frac{e^n}{n^5} x^n$

5. $\sum_{n=1}^{\infty} \frac{1}{n^2} x^n$

6. $\sum_{n=0}^{\infty} \frac{1}{n+2} x^n$

7. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n\, 2^n} x^n$

8. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}} x^n$

9. $\sum_{n=1}^{\infty} \frac{3^n}{5^n \sqrt{n}} x^n$

10. $\sum_{n=0}^{\infty} \frac{e^n}{\sqrt{4^n+1}} x^n$

11. $\sum_{n=0}^{\infty} \frac{2^n}{n!} x^n$

12. $\sum_{n=2}^{\infty} \frac{3^n}{(\ln n)^n} x^n$

13. Suppose that the power series $\sum_{n=0}^{\infty} c_n x^n$ converges for $x = 2$, but diverges for $x = -4$.

(a) Does the series converge when $x = 1$? Explain.

(b) Does the series converge when $x = 5$? Explain.