2.2 Exponential Decay

When the growth constant $k$ is negative, the function

$$y = y_0 e^{kt}$$

do es not actually grow. Instead, the value of $y$ approaches zero as $t \to \infty$. This is known as exponential decay.

When discussing exponential decay, it is common in applications to write $k = -r$, where $r$ is a positive constant known as the decay constant. In this case, the differential equation takes the following form.

**The Exponential Decay Equation**

The exponential decay equation is the differential equation

$$\frac{dy}{dt} = -ry$$

where $r$ is a positive constant. Its solutions have the form

$$y = y_0 e^{-rt}$$

where $y_0 = y(0)$ is the initial value of $y$.

Figure 1 shows the graph of a typical exponential decay. As with exponential growth, there are two alternatives to using the decay constant $r$ when describing exponential decay:

1. The time constant (or e-folding time) of $y$ is the quantity $\tau = 1/r$, and represents the amount of time that it takes for the value of $y$ to be divided by $e$.

2. The half-life of $y$ is the amount of time that it takes for the value of $y$ to be cut in half. It can be found by solving the equation $e^{-rt} = 1/2$ for $t$.

As we shall see, exponential decay can be used to model such diverse phenomena as radioactive decay, electric circuits, and chemical reactions.

**Radioactive Decay**

A substance whose atoms are inherently unstable is called radioactive. For such a substance, a certain fixed proportion of the atoms decay during each time interval. If $N$ is the number of atoms in a sample of the substance, then $N$ will satisfy the differential equation

$$\frac{dN}{dt} = -rN.$$ 

Here $r$ is the decay constant, which represents the rate at which individual atoms tend to decay. For example, if $r = 0.003 \text{ hour}^{-1}$, it means that about 0.3% of the atoms will decay each hour.

The number of atoms in a radioactive sample decays exponentially with time:

$$N = N_0 e^{-rt},$$

where $N_0$ is the number of atoms in the sample at time $t = 0$ (see Figure 2). Equivalently, the total mass $M$ of atoms in a sample will decay exponentially:

$$M = M_0 e^{-rt}.$$
EXAMPLE 1
Cesium-137 has a half-life of approximately 30.17 years. If a 0.300-mole sample of $^{137}\text{Cs}$ is left in a storage closet, how much $^{137}\text{Cs}$ will be left after four years?

SOLUTION The amount $N(t)$ of $^{137}\text{Cs}$ will obey an equation of the form

$$N(t) = 0.300 e^{-rt},$$

where $r$ is a constant. Since the half-life is 30.17 years, we know that

$$e^{-r(30.17)} = \frac{1}{2}.$$ 

Solving for $r$ gives $r \approx 0.022975$/year. Then

$$N(4) = 0.300 e^{-(0.022975)(4)} = 0.273659,$$

so there should be 0.274 moles left after four years.

EXAMPLE 2
A radiochemist prepares a cobalt sample containing 0.100 moles of $^{58}\text{Co}$. According to readings from a Geiger counter, the atoms of $^{58}\text{Co}$ in the sample appear to be decaying at a rate of $6.79 \times 10^{-7}$ moles/min. Based on this information, what is the half-life of $^{58}\text{Co}$?

SOLUTION The amount $N(t)$ of $^{58}\text{Co}$ will obey an equation of the form

$$N(t) = 0.100 e^{-rt},$$

where $r$ is a constant. Then

$$N'(t) = -0.100 r e^{-rt}.$$ 

We are given that $N'(0) = -6.79 \times 10^{-7}$ moles/min, which gives us the equation

$$-6.79 \times 10^{-7} = -0.100 r.$$ 

Solving for $r$ gives $r = 6.79 \times 10^{-6}$/min. The half-life is the value of $t$ for which

$$e^{-rt} = \frac{1}{2}.$$ 

Plugging in $r$ and solving for $t$ yields a half life of 102,084 minutes, which is about $70.9$ days.

RC Circuits
Figure 4 shows a simple kind of electric circuit known as an RC circuit. This circuit has two components:

- A resistor is any circuit component—such as a light bulb—that resists the flow of electric charge. Applying voltage to a resistor will force current through it, with the amount of current given by Ohm’s law

$$I = \frac{V}{R}. \quad (1)$$

Here $V$ is the applied voltage, $I$ is the resulting current, and $R$ is a constant called the resistance of the resistor.
A capacitor is a circuit component that stores electric charge. A charged capacitor can supply voltage to a circuit, with the amount of voltage given by the equation

\[ V = \frac{Q}{C}. \]  

(2)

Here \( Q \) is the charge stored in the capacitor and \( C \) is a constant called the capacitance of the capacitor.

In an RC circuit, the voltage produced by a capacitor is applied directly across a resistor. Substituting equation (2) into equation (1) yields a formula for the resulting current:

\[ I = \frac{Q}{RC}. \]  

(3)

This current represents the flow of charge out of the capacitor, with

\[ \frac{dQ}{dt} = -I. \]

Substituting equation (3) into this equation yields the differential equation

\[ \frac{dQ}{dt} = -\frac{Q}{RC}. \]

Thus the charge \( Q \) will decay exponentially, with decay constant

\[ r = \frac{1}{RC}. \]

In the case where the resistor is a light bulb, the result is that the bulb lights up at first, but becomes dimmer and dimmer over time as the capacitor discharges.

**EXAMPLE 3**

A 0.25 F capacitor holding a charge of 2.0 C is attached to a 1.6 Ω resistor. How long will it take for the capacitor to expend 1.5 C of its initial charge?

**SOLUTION** The charge on the capacitor will decay exponentially according to the formula

\[ Q = Q_0 e^{-rt}, \]

where \( Q_0 = 2.0 \) C and

\[ r = \frac{1}{RC} = \frac{1}{(1.6 \text{ Ω})(0.25 \text{ F})} = 2.5/\text{sec}. \]

If the capacitor expends 1.5 C of its charge, it will have 0.5 C left. Plugging this into the formula for \( Q \) gives

\[ 0.5 = 2.0 e^{-2.5t}, \]

and solving for \( t \) yields \( t = 0.5545 \). Thus the capacitor will expend 1.5 C of charge in approximately 0.55 seconds.

This application assumes you are familiar with the basics of electric circuits, as well as the SI units corresponding to various circuit-related quantities. Here is a summary:

- Electric charge is a basic property of matter, inherent to charged particles such as electrons and protons. It is measured in coulombs (C).
- Voltage is a difference in electric potential that causes charge to move through a circuit. It is measured in volts (V).
- Electric current is the rate at which electric charge flows through a circuit. It is measured in amps (A), where 1 A = 1 C/sec.
- Resistance is a property of resistors. It is measured in ohms (Ω), where 1 Ω = 1 V/A.
- Capacitance is a property of capacitors. It is measured in farads (F), where 1 F = 1 C/V.

Equivalently, the time constant for the RC circuit is

\[ \tau = RC. \]
EXERCISES

1. A sample of an unknown radioactive isotope initially weighs 5.00 g. One year later the mass has decreased to 4.27 g.
   (a) How quickly is the mass of the isotope decreasing at that time?
   (b) What is the half life of the isotope?

2. During a certain chemical reaction, the concentration $[C_4H_9Cl]$ of butyl chloride obeys the rate equation
   \[ \frac{d[C_4H_9Cl]}{dt} = -r[C_4H_9Cl], \]
   where $r = 0.1223/\text{sec}$ is the rate constant for the reaction. How long will it take for this reaction to consume 90% of the initial butyl chloride?

3. A capacitor with a capacitance of 5.0 F holds an initial charge of 350 C. The capacitor is attached to a resistor with a resistance of 9.0 Ω.
   (a) How quickly will the charge held by the capacitor initially decrease?
   (b) How quickly will the charge be decreasing after 20 seconds?

4. An LR circuit consists of a resistor attached to an electrical component called an inductor, which supplies voltage to the resistor according to the formula
   \[ V = -L \frac{dI}{dt}. \]
   Here $L$ is a constant called the inductance of the inductor.
   (a) Combine the equation above with Ohm’s law to obtain a differential equation for the current $I(t)$ that involves the constants $L$ and $R$.
   (b) The current $I(t)$ in an LR circuit decays exponentially. Find a formula for the decay constant in terms of $L$ and $R$.

5. For a planetary atmosphere of ideal gas of uniform temperature $T$, the atmospheric pressure $P(h)$ and density $\rho(h)$ at a height $h$ above the ground are related by the equations
   \[ P = \rho RT \quad \text{and} \quad \frac{dP}{dh} = -\rho g, \]
   where $R$ is the specific gas constant and $g$ is the acceleration due to gravity.
   (a) Combine the given equations to obtain a single differential equation for $P$ involving the constants $g$, $R$, and $T$.
   (b) The atmospheric pressure in such an atmosphere varies with height according to the formula $P(h) = P_0 e^{-rh}$, where $P_0$ is the pressure at ground level. Find a formula for the decay constant $r$ in terms of $g$, $R$, and $T$. 

The first of these equations is the molar form of the ideal gas law, while the second is the equation for hydrostatic pressure.