8.1 Higher Dimensions

It is time for us to tackle the idea of \(n\)-dimensional space a little more directly. Here **\(n\)-dimensional space** refers to a geometric space \(\mathbb{R}^n\) with \(n\) spatial dimensions, where \(n\) can be any positive integer. For example, \(\mathbb{R}^1\) is an infinite line, \(\mathbb{R}^2\) is an infinite plane, and \(\mathbb{R}^3\) is a three-dimensional space that is infinite in all directions. When \(n \geq 4\), the space \(\mathbb{R}^n\) is said to be **higher-dimensional**.

Before we discuss the mathematics of higher-dimensional spaces, a few words about philosophy are in order. There is a basic philosophical objection to higher-dimensional spaces, which is that there are only three dimensions in the physical world. What does it even mean to discuss the geometry of four or five-dimensional space if these spaces don’t really exist?

The answer is that we don’t need these spaces to exist physically to be able to talk about them. Four and five-dimensional spaces exist on the same level as other mathematical objects, such as the number 10, the function \(f(x) = x^2\), or the interval \([-1, 1]\). None of these things have any real physical existence—they are abstractions, which exist in the sense that they refer to certain aspects of real things. Thus we can have ten books, or the temperature can be ten degrees, but the number 10 itself isn’t real in any physical sense.

What four-dimensional space refers to is the set of possibilities for a system that can be described by four real variables. For example, if a chemical reaction involves four different reactants, then the concentrations \((C_1, C_2, C_3, C_4)\) of the reactants are an ordered quadruple of real numbers. If a sector of the economy involves four goods, then the prices \((p_1, p_2, p_3, p_4)\) of the goods are an ordered quadruple of real numbers. In each case, the set of all possible values for this quadruple can be thought of as a four-dimensional space, with each specific quadruple being a point in this space.

The reason we refer to \(\mathbb{R}^n\) as a “space” is that we would like to extend our geometric intuition for \(\mathbb{R}^2\) and \(\mathbb{R}^3\) to higher dimensions as much as possible. It turns out that \(\mathbb{R}^n\) is similar enough to \(\mathbb{R}^2\) and \(\mathbb{R}^3\) that it helps to think about it in geometric terms. But when we refer to a quadruple such as \((5, 3, 2, 7)\) as a “point” in \(\mathbb{R}^4\), we are really just making an analogy to points in \(\mathbb{R}^2\) and \(\mathbb{R}^3\). Because higher-dimensional spaces only exist in the abstract, we must always be very careful to define geometric terms precisely before using them in this context. For example, the term “distance” seems self-explanatory in two and three dimensions, but for higher dimensions we must say exactly what we mean by “distance” before we can use this concept.

Thus, our description of higher dimensions will include precise definitions of many basic geometric concepts. In most cases, these definitions will be based on the descriptions of these concepts that we obtained in \(\mathbb{R}^2\) and \(\mathbb{R}^3\). For example, the Pythagorean theorem is a theorem of Euclidean plane geometry, but in higher dimensions it becomes part of the definition of distance.

**Points and Coordinates in \(\mathbb{R}^n\)**

A **point** in \(n\) dimensions is simply a list of \(n\) real numbers. For example, \((5, 3, 2, 7)\) is a point in four dimensions, and \((8, -1, 3, 0, 9, 1/2)\) is a point in six dimensions. The individual numbers are called the **coordinates** of the point. The set of all points with \(n\) coordinates is denoted \(\mathbb{R}^n\), and is referred to as **\(n\)-dimensional Euclidean space** or simply **\(n\)-dimensional space**.

In \(\mathbb{R}^3\) the three coordinates of a point are usually called \(x, y,\) and \(z\). When working with four or more dimensions, though, it is too cumbersome to use a different letter for each coordinate. Instead, we refer to the \(n\) coordinates in \(\mathbb{R}^n\) as \(x_1, x_2,\) and so forth.
with the last coordinate being $x_n$. For example, the point 

$$(5, 3, 2, 7)$$

has 5 as its $x_1$-coordinate, 3 as its $x_2$-coordinate, 2 as its $x_3$-coordinate, and 7 as its $x_4$-coordinate.

Vectors in $\mathbb{R}^n$

A vector in $\mathbb{R}^n$ is a column of $n$ real numbers:

$$
\mathbf{v} = \begin{bmatrix}
v_1 \\
v_2 \\
\vdots \\
v_n
\end{bmatrix}
$$

The numbers $v_1, v_2, \ldots, v_n$ are called the components of the vector. As in $\mathbb{R}^2$ and $\mathbb{R}^3$, it is convenient to think of vectors and points in $\mathbb{R}^n$ as being the same thing:

$$
\mathbf{v} = (v_1, v_2, \ldots, v_n) = \begin{bmatrix}
v_1 \\
v_2 \\
\vdots \\
v_n
\end{bmatrix}
$$

In $\mathbb{R}^2$ and $\mathbb{R}^3$, we defined addition of vectors geometrically using arrows, and then worked out that it corresponds to the componentwise sum. For higher dimensions, though, we use componentwise sum as the definition of addition.

$$
\begin{bmatrix}
v_1 \\
v_2 \\
\vdots \\
v_n
\end{bmatrix} + 
\begin{bmatrix}
w_1 \\
w_2 \\
\vdots \\
w_n
\end{bmatrix} = 
\begin{bmatrix}
v_1 + w_1 \\
v_2 + w_2 \\
\vdots \\
v_n + w_n
\end{bmatrix}
$$

We imagine vector addition as having the same geometric meaning in higher dimensions that it does in $\mathbb{R}^2$ and $\mathbb{R}^3$, as shown in Figure 1.

Scalar multiplication is also defined componentwise:

$$
\begin{bmatrix}
v_1 \\
v_2 \\
\vdots \\
v_n
\end{bmatrix} k = 
\begin{bmatrix}
k v_1 \\
k v_2 \\
\vdots \\
k v_n
\end{bmatrix}
$$

Again, we imagine this as having the same geometric meaning that it does in $\mathbb{R}^2$ and $\mathbb{R}^3$. For example, multiplying a vector by 3 should increase its length by a factor of 3 without changing this direction, and multiplying a vector by $-2$ should double its length and make it point in the opposite direction.

There are $n$ different standard basis vectors in $\mathbb{R}^n$, which we denote $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$.

$$
\mathbf{e}_1 = 
\begin{bmatrix}
1 \\
0 \\
\vdots \\
0
\end{bmatrix}, 
\mathbf{e}_2 = 
\begin{bmatrix}
0 \\
1 \\
\vdots \\
0
\end{bmatrix}, 
\ldots, 
\mathbf{e}_n = 
\begin{bmatrix}
0 \\
0 \\
\vdots \\
1
\end{bmatrix}
$$
Any vector $v$ in $\mathbb{R}^n$ can be written as a linear combination of the standard basis vectors:

$$v = v_1 e_1 + v_2 e_2 + \cdots + v_n e_n.$$ 

### Magnitude, Distances, and Angles

The **magnitude** of a vector $v$ in $\mathbb{R}^n$ is defined by the formula

$$|v| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

If we imagine vectors as arrows in $\mathbb{R}^n$, then the magnitude can be thought of as the length of the arrow.

**EXAMPLE 1**

Find the magnitude of the vector $(2, 4, 2, 5)$.

**SOLUTION**  We have

$$|(2, 4, 2, 5)| = \sqrt{2^2 + 4^2 + 2^2 + 5^2} = \sqrt{49} = 7$$

If $p$ and $q$ are points in $\mathbb{R}^n$, the **distance** from $p$ to $q$ is defined by the formula

$$\text{distance}(p, q) = |p - q| = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2 + \cdots + (p_n - q_n)^2}$$

Note then that the magnitude $|p|$ of a point $p$ represents its distance from the origin.

**EXAMPLE 2**

Find the distance between the points $(8, 7, 0, 3)$ and $(3, 1, 4, 1)$ in $\mathbb{R}^4$.

**SOLUTION**  The distance is

$$|(8, 7, 0, 3) - (3, 1, 4, 1)| = |(5, 6, -4, 2)| = \sqrt{5^2 + 6^2 + (-4)^2 + 2^2} = 9$$

The **dot product** of two vectors $v, w$ in $\mathbb{R}^n$ is defined by the formula

$$v \cdot w = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$$

We can use dot product to define the angle between vectors. If $v$ and $w$ are nonzero vectors in $\mathbb{R}^n$, the **angle between $v$ and $w$** is defined to be the value of $\theta$ between $0^\circ$ and $180^\circ$ that satisfies the equation

$$v \cdot w = |v| |w| \cos \theta$$

We say that $v$ and $w$ are **orthogonal** if $v \cdot w = 0$. Two nonzero vectors are orthogonal if and only if the angle between them is $90^\circ$. 

Though cross product as such only makes sense in $\mathbb{R}^3$, there is a nice generalization of cross product to any number of dimensions. This operation takes $n - 1$ vectors in $\mathbb{R}^n$ as input, and outputs a new vector that is orthogonal to all of them. For example, if $u$, $v$, and $w$ are vectors in $\mathbb{R}^4$, then the determinant

$$
\begin{vmatrix}
e_1 & e_2 & e_3 & e_4 \\
u_1 & u_2 & u_3 & u_4 \\
v_1 & v_2 & v_3 & v_4 \\
w_1 & w_2 & w_3 & w_4 \\
\end{vmatrix}
$$

yields a vector $c$ in $\mathbb{R}^4$ that is orthogonal to $u$, $v$, and $w$. Moreover, the magnitude of $c$ is precisely the volume of the (3-dimensional) parallelepiped in $\mathbb{R}^4$ determined by $u$, $v$, and $w$.

Interestingly, in the case of $\mathbb{R}^2$ this generalized cross product takes only a single vector $v$ as input, and is given by the formula

$$
\begin{vmatrix}
i & j \\
v_x & v_y \\
\end{vmatrix} = (v_y, -v_x).
$$

The resulting vector has the same magnitude as $v$, but is turned $90^\circ$ clockwise. Thus the operation of turning a vector $90^\circ$ can be thought of as a two-dimensional analog of cross product!

### EXAMPLE 3

Find the angle between the vectors $(2, 3, 4, 5)$ and $(3, 1, 2, 2)$ in $\mathbb{R}^4$.

**SOLUTION**  Let $v = (2, 3, 4, 5)$ and $w = (3, 1, 2, 2)$. Then the equation

$$
v \cdot w = |v||w| \cos \theta
$$

becomes

$$
27 = \sqrt{54} \sqrt{18} \cos \theta.
$$

Solving for $\cos \theta$ and simplifying yields $\cos \theta = \sqrt{3}/2$, and therefore $\theta = 30^\circ$.

**Geometry in $\mathbb{R}^n$**

Essentially all of the geometry that we know in $\mathbb{R}^2$ and $\mathbb{R}^3$ continues to work in $\mathbb{R}^n$, assuming we interpret all of the geometric terms correctly using vectors. For example, four points $a, b, c, d$ in $\mathbb{R}^n$ are said to be the vertices of a parallelogram if

$$
c - d = b - a,
$$
as shown in Figure 2. Such a parallelogram is called a **rectangle** if $d - a$ is orthogonal to $b - a$, as shown in Figure 3, and a rectangle is called a **square** if $|d - a| = |b - a|$.

The reader should tentatively assume that all geometric concepts from two and three dimensions continue to make sense in higher dimensions, and that these concepts interact in all of the familiar ways. For example, it makes perfect sense to talk about a circle in $\mathbb{R}^n$, and the area of such a circle is still $\pi r^2$, where $r$ is the radius. Lines and planes make also make sense in $\mathbb{R}^n$, and so forth. Essentially all of the vector geometry we have learned in $\mathbb{R}^2$ and $\mathbb{R}^3$ continues to work for these same shapes in $\mathbb{R}^n$. 

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**A Closer Look**

**Cross Product in $\mathbb{R}^n$**

A similar formula works in $\mathbb{R}^n$, with the standard basis vectors $e_1, e_2, \ldots, e_n$ on the first row of an $n \times n$ determinant, and the $n - 1$ input vectors on the remaining rows.
EXAMPLE 4

Figure 4 shows a square in $\mathbb{R}^4$. Find the coordinates of the point $p$.

**SOLUTION**  Let $v$ and $w$ be the parallel vectors shown in Figure 5. We can find $w$ by subtracting the endpoints:

$$w = (9,1,5,4) - (6,4,8,1) = (3,-3,-3,3).$$

This gives us the direction of $v$. The magnitude of $v$ is the side length of the square, which is the distance between the two bottom points:

$$|v| = \sqrt{5^2 + (-5)^2 + 5^2 + (-5)^2} = 10.$$

Thus $v$ is parallel to $w$ but has a magnitude of 10. Since $\frac{|w|}{\sqrt{3^2 + (-3)^2 + 3^2 + (-3)^2}} = 6$,

we conclude that $v = (10/6)w = (5/3)w = (5,-5,-5,5)$. Then

$$p = (1,9,3,6) + v = (1,9,3,6) + (5,-5,-5,5) = (6,4,-2,11).$$

Of course, not all of the geometry in higher dimensions is so mundane. In addition to two and three dimensional shapes, there are also lots of interesting high-dimensional shapes in $\mathbb{R}^n$, including hyperspheres (higher-dimensional analogs of circles and spheres) and hypercubes (higher-dimensional analogs of squares and cubes), but we must learn quite a bit of vector geometry and vector calculus in $\mathbb{R}^n$ before we can investigate the properties of such shapes.

**EXERCISES**

1. Find the distance between the points $(5,1,3,7,6)$ and $(4,2,6,9,5)$ in $\mathbb{R}^5$.

2. Find the angle between the vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ -1 \\ 5 \\ 3 \end{bmatrix}$ in $\mathbb{R}^4$.

3. The following figure shows a trapezoid in $\mathbb{R}^4$.

![Trapezoid in $\mathbb{R}^4$](image)

Find the coordinates of the point $p$.

4. Find the area of the triangle in $\mathbb{R}^4$ with vertices $(1,1,0,0)$, $(0,1,1,0)$, and $(0,0,1,1)$. 
5. The following figure shows a right triangle in $\mathbb{R}^4$.

Find the coordinates of the point $p$. 