1.4 Initial Value Problems

As we have seen, most differential equations have more than one solution. For a first-order equation, the general solution usually involves an arbitrary constant $C$, with one particular solution corresponding to each value of $C$.

What this means is that knowing a differential equation that a function $y(x)$ satisfies is not enough information to determine $y(x)$. To find the formula for $y(x)$ precisely, we need one more piece of information, usually called an initial condition.

For example, suppose we know that a function $y(x)$ satisfies the differential equation

$$y' = y.$$ 

It follows that

$$y(x) = Ce^x$$

for some constant $C$. If we want to determine $C$, we need at least one more piece of information about the function $y(x)$. For example, if we also know that

$$y(0) = 3,$$

the value of $C$ must be 3, and hence $y(x) = 3e^x$.

**Initial Value Problems**

An initial value problem consists of

1. A first-order differential equation $y' = f(x, y)$, and
2. An initial condition of the form $y(a) = b$.

For example,

$$y' = y, \quad y(0) = 3$$

is an initial value problem, whose solution is

$$y = 3e^x.$$ 

In general, we expect that every initial value problem has exactly one solution. We can find this solution using the following procedure.

**Solving Initial Value Problems**

Given an initial value problem

$$y' = f(x, y), \quad y(a) = b,$$

we can solve it using the following procedure:

1. Find the general solution to the given differential equation, involving an arbitrary constant $C$.
2. Substitute $x = a$ and $y = b$ to get an equation for $C$.
3. Solve for $C$ and then substitute the answer back into the formula for $y$. 

EXAMPLE 1
Find the solution to the following initial value problem:

\[ y' = -y^2, \quad y(0) = 5. \]

**SOLUTION** We previously found the general solution to this differential equation:

\[ y = \frac{1}{x + C}. \]

Plugging in \( x = 0 \) and \( y = 5 \) gives the equation

\[ 5 = \frac{1}{0 + C}. \]

Solving for \( C \) gives \( C = 1/5 \), so

\[ y = \frac{1}{x + (1/5)}. \]

This simplifies to

\[ y = \frac{5}{5x + 1}. \]

In this last step we multiplied the numerator and denominator by 5 to simplify the fraction of fractions.

EXAMPLE 2
Find the solution to the following initial value problem:

\[ y' = 2y, \quad y(0) = 5. \]

**SOLUTION** The given differential equation isn’t very different from the equation

\[ y' = y. \]

In that case, the general solution was \( y = Ce^x \). How can we modify this solution to account for the extra 2?

A few moments of thought reveals the answer:

\[ y = Ce^{2x}. \]

So this is the general solution to the given equation. Plugging in \( x = 0 \) and \( y = 5 \) gives the equation

\[ 5 = Ce^0, \]

so \( C = 5 \) and the solution is

\[ y = 5e^{2x}. \]

More generally, the solution to any equation of the form \( y' = ky \) (where \( k \) is a constant) is \( y = Ce^{kx} \).

The Fundamental Theorem of ODE’s (Optional)
As a general rule, we expect any initial value problem of the form

\[ y' = f(x, y), \quad y(a) = b \]

to have a unique solution. The following theorem gives specific conditions which guarantee that this holds.
Fundamental Theorem of ODE's
Consider an initial value problem of the form
\[ y' = f(x, y), \quad y(a) = b. \]
If the function \( f(x, y) \) is continuously differentiable for all values of \( x \) and \( y \), then this initial value problem has a unique solution.

This theorem is also known as the **existence and uniqueness theorem for first-order ODE's**, since it guarantees both that the solution exists and that it is unique.

The hypothesis that the function \( f(x, y) \) is continuously differentiable is important for the theorem. In fact, there are initial value problems that do not satisfy this hypothesis that have more than one solution. For example, the initial value problem
\[ y' = \frac{y}{x}, \quad y(0) = 0 \]
has infinitely many different solutions, namely the lines \( y = Cx \) for all possible values of \( C \). The function \( f(x, y) \) in this case is \( y/x \), which is not defined (and hence not continuously differentiable) when \( x = 0 \).

There is a nice geometric interpretation of the fundamental theorem. As we have seen, the solutions to a differential equation can be viewed as a family of solution curves in the \( xy \)-plane. For example, Figure 1 shows the curves \( y = \ln(x + C) \), which are the solutions to the differential equation
\[ y' = e^{-y}. \]
From a geometric point of view, an initial condition \( y(a) = b \) is the same as a point \((a, b)\) that the solution curve must pass through. Thus, saying that the initial value problem
\[ y' = f(x, y), \quad y(a) = b \]
has a unique solution is the same as saying that the point \((a, b)\) has exactly one solution curve passing through it. This leads us to the following restatement of the fundamental theorem of ODE's.

\[ \text{Figure 1:} \ (a) \text{ Three curves of the form } y = \ln(x + C). \ (b) \text{ The family of all such curves completely fills the plane.} \]
Fundamental Theorem of ODE’s (Geometric Version)
Consider a first-order differential equation of the form
\[ y' = f(x, y), \]
where the function \( f(x, y) \) is continuously differentiable. Then:
1. The solution curves for this differential equation completely fill the plane, and
2. Solution curves for different solutions do not intersect.

Here statement (1) is the same as saying that every point \((a, b)\) lies on at least one solution curve, i.e. every initial condition gives at least one solution. Statement (2) is the same as saying that no point \((a, b)\) lies on more than one solution curve, i.e. every initial condition has at most one solution.

EXERCISES
1–2 ■ Solve the given initial value problem.

1. \( y' = xe^x, \ y(0) = 3 \)
2. \( y' = 3y, \ y(2) = 4 \)