11.1 Linear Systems

Recall that a linear equation is any equation of the form

\[ a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b \]

where \( x_1, x_2, \ldots, x_n \) are variables and \( a_1, a_2, \ldots, a_n \) and \( b \) are constants. A linear system is a system of linear equations, i.e. a collection of linear equations involving the same variables. For example,

\[
\begin{align*}
5x + 3y - 4z &= 15 \\
4x - 8y + 5z &= 18
\end{align*}
\]

is a linear system. Specifically, this is a \( 2 \times 3 \) system, meaning that it involves 2 equations and 3 variables.

A solution to a linear system is an assignment of values to the variables that makes all of the equations true. For example, \( x = 4, y = 1, z = 2 \) is a solution to the \( 2 \times 3 \) system above, since substituting these values in for the variables makes both equations true. We can think of this solution as an ordered triple \((4, 1, 2)\), i.e. a point in \( \mathbb{R}^3 \).

The set of all possible solutions to a linear system is called its solution set. For a linear system with \( n \) variables, the solution set is a set of points in \( \mathbb{R}^n \). For example, the solution set for the \( 2 \times 3 \) linear system above is a line in \( \mathbb{R}^3 \).

\( 2 \times 2 \) Systems

For a linear system in two variables, each equation represents a line in \( \mathbb{R}^2 \). Thus a \( 2 \times 2 \) system such as

\[
\begin{align*}
x + 4y &= 11 \\
2x - 3y &= 0
\end{align*}
\]

represents two lines in \( \mathbb{R}^2 \), as shown in Figure 1. There is only one solution to this system, namely the point \((3, 2)\), since this is the only point that lies on both of the lines. This behavior is typical: a \( 2 \times 2 \) system usually has exactly one solution.

There are exceptions to this rule. For example, the system

\[
\begin{align*}
x + 2y &= 1 \\
2x + 4y &= 5
\end{align*}
\]

has no solutions, for multiplying the first equation through by 2 gives \( 2x + 4y = 2 \), which contradicts the second equation. Geometrically, these two equations correspond to parallel lines in \( \mathbb{R}^2 \), as shown in Figure 2.

Let’s see what happens if we try to solve the system in the usual way. Solving for \( x \) in the first equation gives

\[ x = 1 - 2y \]

and plugging this into the second equation gives

\[ 2(1 - 2y) + 4y = 5. \]

The \( y \) terms on the left side cancel, leaving the contradictory equation \( 2 = 5 \), which indicates the contradiction in the original linear system.
3 × 2 Systems

Now consider a $3 \times 2$ system such as

$x + 4y = 16$
$3x - 4y = 0$
$6x + 2y = 16$

This system corresponds to three lines in $\mathbb{R}^2$, as shown in Figure 3. Because there aren’t any points that lie on all three lines, this system has no solutions at all! Again, this behavior is typical: a $3 \times 2$ system usually has no solutions.

Of course, it is possible for a $3 \times 2$ system to have a solution, if all three lines happen to go through the same point. For example, if we change the equation $6x + 2y = 11$ to $6x + 2y = 30$, then all three lines go through the point $(4, 3)$, as shown in Figure 4. Thus the $3 \times 2$ system

$x + 4y = 16$
$3x - 4y = 0$
$6x + 2y = 30$

has exactly one solution, namely the point $(4, 3)$.

In general, the easiest way to check whether a $3 \times 2$ system has a solution is to find the intersection point of the first two lines, and then check whether this point also lies in the third line. If it does, then the point is a solution to the system. If it doesn’t, then the system has no solutions.

**EXAMPLE 1**

Determine whether the following $3 \times 2$ system has a solution:

$5x + 2y = 18$
$3x - 4y = 16$
$2x + 3y = 9$

**SOLUTION**

The first two equations form a $2 \times 2$ system:

$5x + 2y = 18$
$3x - 4y = 16$

Solving this system in the usual way, we find that $x = 4$ and $y = -1$. Plugging this into the third equation gives

$2(4) + 3(-1) = 9$,

which isn’t true, so this system doesn’t have a solution.

Linear Systems in $\mathbb{R}^3$

For a linear system with three variables, each linear equation represents a plane in $\mathbb{R}^3$. Thus a $3 \times 3$ system such as

$x + 9y - z = 27$
$11x - 5z = 12$
$2x + y + 15z = 37$
corresponds to three planes in $\mathbb{R}^3$, as shown in Figure 5. There is only one point that lies on all three planes, namely $(2, 3, 2)$, and this is the unique solution to the system. Again, this behavior is typical: a $3 \times 3$ system usually has exactly one solution, since three planes in $\mathbb{R}^3$ typically intersect at a single point.

As with the two-variable case, a three-variable system with extra equations usually has no solution. For example, a $4 \times 3$ system usually doesn’t have a solution, since four planes in $\mathbb{R}^3$ typically don’t have a point in common. Of course, it is possible for a $4 \times 3$ system to have a solution, if the fourth plane happens to go through the point of intersection of the first three planes.

Finally, something interesting happens if we have fewer than three equations. For example, the $2 \times 3$ system
\[
\begin{align*}
-x_1 + 4x_2 - 7x_3 + 3x_4 - 9x_5 + 2x_6 &= 3 \\
-4x_1 + 3x_2 + 5x_3 + x_4 - 8x_5 - 9x_6 &= 12 \\
x_1 - 3x_2 - 9x_4 + 6x_5 + 8x_6 &= -47 \\
7x_1 + 9x_2 - 7x_3 - x_4 - 4x_5 + 2x_6 &= 11
\end{align*}
\]
consists of two planes in $\mathbb{R}^3$, as shown in Figure 6. These two planes intersect along a line, and every point on this line is a solution to the system. In general, a $2 \times 3$ system usually has infinitely many solutions, since two planes in $\mathbb{R}^3$ typically intersect along a line. Of course, this doesn’t happen when the two planes are parallel, in which case the system has no solutions.

### Linear Systems in $\mathbb{R}^n$

In applications of mathematics, linear systems often have a very large number of variables. For example, the PageRank algorithm, which is used by Google to order search engine results, involves a system of linear equations with tens of billions of variables. One of the primary goals of linear algebra is to understand the behavior of such large systems and to develop feasible methods for solving them.

Here is an example of a $4 \times 6$ linear system:
\[
\begin{align*}
-x_1 + 4x_2 - 7x_3 + 3x_4 - 9x_5 + 2x_6 &= 3 \\
-4x_1 + 3x_2 + 5x_3 + x_4 - 8x_5 - 9x_6 &= 12 \\
x_1 - 3x_2 - 9x_4 + 6x_5 + 8x_6 &= -47 \\
7x_1 + 9x_2 - 7x_3 - x_4 - 4x_5 + 2x_6 &= 11
\end{align*}
\]
Here the variables are $x_1, x_2, x_3, x_4, x_5,$ and $x_6$. To increase readability, we have aligned the terms of the equations into columns, with one column on the left for each variable, and a final column on the right for the constant terms.

In general, each linear equation in $n$ variables defines a hyperplane in $\mathbb{R}^n$, i.e. a flat of dimension $n - 1$. For example, a linear equation in six variables defines a hyperplane in $\mathbb{R}^6$, which is a five-dimensional flat. The solution set to a linear system is the intersection of all of the corresponding hyperplanes.

Certain features of linear systems that we have observed for two and three variables carry over to the general case.

#### Number of Solutions to a Linear System

A linear system always has either
1. No solutions,
2. One solution, or
3. Infinitely many solutions.
The number of solutions is usually determined by the number of equations.

**Types of Linear Systems**

1. A linear system with fewer equations than variables is called **underdetermined**. Such a system usually has infinitely many different solutions.
2. A linear system with the same number of equations as variables is called **square**. Such a system usually has exactly one solution.
3. A linear system with more equations than variables is called **overdetermined**. Such a system usually has no solutions.

**EXERCISES**

1–4 Determine whether the given linear system has no solutions, one solution, or infinitely many solutions.

1. \[ \begin{align*}
4x + 6y &= 5 \\
6x + 9y &= 8
\end{align*} \]
2. \[ \begin{align*}
2x - 3y &= 1 \\
3x - 4y &= 3
\end{align*} \]
3. \[ \begin{align*}
3x + 2y - 4z &= 5 \\
2x - 3y + 5z &= 8
\end{align*} \]
4. \[ \begin{align*}
x + 2y - 3z &= 4 \\
2x + 4y - 6z &= 5
\end{align*} \]

5. For what value of \( a \) does the following linear system have a solution?

\[ \begin{align*}
2x + 3y &= 11 \\
x + 4y &= 8 \\
3x - 2y &= a
\end{align*} \]

6. For what value of \( a \) does the following linear system have no solutions?

\[ \begin{align*}
3x + 2y + 4z &= 7 \\
6x + 4y + az &= 3
\end{align*} \]