11.3 No Solutions

Linear systems sometimes have no solutions at all. For example, the system

\[ \begin{align*}
2x + 4y &= 10 \\
3x + 6y &= 17
\end{align*} \]

has no solution, since the corresponding lines are parallel in \( \mathbb{R}^2 \). Algebraically, if \( 2x + 4y = 10 \), then \( 3x + 6y \) must be 15, not 17.

Here are the first few steps of the row reduction for the above system:

\[
\begin{bmatrix}
2 & 4 & 10 \\
3 & 6 & 17
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 5 \\
3 & 6 & 17
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 5 \\
0 & 0 & 2
\end{bmatrix}
\]

At this point, the equation corresponding to the second row is

\[ 0x + 0y = 2 \]

or more succinctly

\[ 0 = 2 \]

which is a contradiction.

In general, a row whose coefficients are all zero but whose constant term is nonzero indicates a contradiction. If such a row arises during a row reduction, it means that the original linear system had no solutions.

3 × 3 Systems with No Solutions

It is easy to see when a 2 × 2 system has no solutions, since the two lines must be parallel. For a 3 × 3 system, though, a contradiction can be much less obvious, and can involve all three of the equations. Geometrically, this corresponds to the situation shown in Figure 1. The three planes in this figure have no point in common, even though no two of the planes are parallel.

An example of this phenomenon is the system

\[ \begin{align*}
2x + 4y + 4z &= 2 \\
3x + 4y + 2z &= 5 \\
5x + 8y + 6z &= 4
\end{align*} \]

Even though no two of these planes are parallel, this 3 × 3 system has no solutions. The reason is that the sum of the first two equations is

\[ 5x + 8y + 6z = 7 \]

which contradicts the third equation.

More generally, a 3×3 system will have no solutions if the third equation contradicts any linear combination of the first two. For example, the linear system

\[ \begin{align*}
2x + 6y - 4z &= 2 \\
x - 5y + 5z &= 5 \\
7x - 3y + 7z &= 6
\end{align*} \]

has no solutions, and the reason is that two times the first equation plus three times the second equation is

\[ 7x - 3y + 7z = 19 \]

which contradicts the third equation.
The following example illustrates how to use row reduction to detect a contradiction in a $3 \times 3$ system.

**EXAMPLE 1**

Solve the following linear system.

\[
\begin{align*}
2x + 4y + 4z &= 2 \\
3x + 4y + 2z &= 5 \\
5x + 8y + 6z &= 4
\end{align*}
\]

**SOLUTION** We row reduce the matrix in the usual way:

\[
\begin{bmatrix}
2 & 4 & 4 & 2 \\
3 & 4 & 2 & 5 \\
5 & 8 & 6 & 4
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 2 & 1 \\
3 & 4 & 2 & 5 \\
5 & 8 & 6 & 4
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 2 & 1 \\
0 & -2 & -4 & 2 \\
5 & 8 & 6 & 4
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 2 & 1 \\
0 & -2 & -4 & -1 \\
0 & -2 & -4 & -1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 2 & 1 \\
0 & 1 & 2 & -1 \\
0 & 0 & 0 & -3
\end{bmatrix}
\]

We can stop the row reduction at this point, since the last row is a contradiction ($0 = -3$). This means that the original linear system had no solutions.

**Overdetermined Systems**

As we have seen, a linear system with more equations than unknowns usually has no solutions. Again, the reason is always a contradiction in the original equations. For example, the system

\[
\begin{align*}
x + 3y &= 2 \\
2x + 3y &= 1 \\
5x + 9y &= 3
\end{align*}
\]

has no solution, and the reason is that the first equation plus twice the second equation is

\[5x + 9y = 4\]

which contradicts the third equation. This contradiction can easily be detected using row reduction:

\[
\begin{bmatrix}
1 & 3 & 2 \\
2 & 3 & 1 \\
5 & 9 & 3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 3 & 2 \\
0 & -3 & -3 \\
0 & -6 & -7
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 3 & 2 \\
0 & 1 & 1 \\
0 & -6 & -7
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 3 & 2 \\
0 & 1 & 1 \\
0 & 0 & -1
\end{bmatrix}
\]

The third row is now the equation $0 = -1$, which is a contradiction.

Of course, it's possible for an overdetermined system to have a solution. For example, the linear system

\[
\begin{align*}
x + 3y &= 2 \\
2x + 3y &= 1 \\
5x + 9y &= 4
\end{align*}
\]
has \((-1, 1)\) as a solution. In this case, the third equation is a consequence of the first two equations. Specifically, the third equation is equal to the first equation plus twice the second equation.

Here is the corresponding row reduction:

\[
\begin{bmatrix}
1 & 3 & 2 \\
2 & 3 & 1 \\
5 & 9 & 4
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 3 & 2 \\
0 & -3 & -3 \\
0 & -6 & -6
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

This time there is no contradiction in the third row, since the third equation is \(0 = 0\).

In general, a row of zeroes obtained during row reduction indicates that one of the original equations was a consequence of the others. If such a row arises during a row reduction, the proper procedure is to move it to the bottom of the matrix and ignore it for the rest of the reduction.

**EXERCISES**

1–2 ■ Use row reduction to solve the given linear system.

1. \(x + 2y + 3z = 4\)
   \(2x + y + 9z = 8\)
   \(x + 4y + z = 10\)

2. \(3x + 12y = 6\)
   \(5x + 11y = 1\)
   \(7x + 10y = -4\)

3–8 ■ Solve the linear system corresponding to the given matrix.

3. \[
\begin{bmatrix}
1 & 3 \\
4 & 8 \\
1 & 7
\end{bmatrix}
\]

4. \[
\begin{bmatrix}
2 & -4 \\
-3 & 7 \\
1 & -2
\end{bmatrix}
\]

5. \[
\begin{bmatrix}
2 & -6 \\
-3 & 9 \\
2 & -4 \\
-5 & 11 \\
4 & -9
\end{bmatrix}
\]

6. \[
\begin{bmatrix}
-2 & -6 \\
-2 & -8 \\
-1 & -7 \\
-2 & -7
\end{bmatrix}
\]

[2 4]

[2 -6 4]

[1 5 2]

[4 -9 -1]

[10 -9 5 4]

[10 -9 5]

[4 -9 -1]