Linear Dependence Tests

The book omits a few key tests for checking the linear dependence of vectors. These short notes discuss these tests, as well as the reasoning behind them.

Our first test checks for linear dependence of the rows of a matrix. It is essentially the same as the algorithm we have been using to test for redundancy in a system of linear equations:

**Theorem 1 Linear Dependence Test for Rows**

Consider a $k \times n$ matrix

\[
A = \begin{bmatrix}
\mathbf{r}_1 \\
\mathbf{r}_2 \\
\vdots \\
\mathbf{r}_k
\end{bmatrix}
\]

whose rows are the row vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_k$. Let $E$ be an echelon matrix obtained by reducing $A$. Then $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_k$ are linearly dependent if and only if $E$ has a row of zeroes.

Given any vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, we can use this theorem to test for linear dependence as follows:

1. First, construct a matrix $A$ whose rows are $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ (or, more precisely, whose rows are the row vectors $\mathbf{v}_1^T, \mathbf{v}_2^T, \ldots, \mathbf{v}_k^T$).

2. Next, use elementary row operations to reduce $A$ to an echelon matrix $E$.

3. Then the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly dependent if and only if $E$ has a row of zeroes.
EXAMPLE 1  Determine whether the three vectors \( \mathbf{u} = (1, 2, 3, 2), \mathbf{v} = (2, 5, 5, 5), \) and \( \mathbf{w} = (2, 6, 4, 6) \) are linearly dependent.

SOLUTION  We begin by constructing a matrix whose rows are \( \mathbf{u}^T, \mathbf{v}^T, \) and \( \mathbf{w}^T: \)

\[
\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 5 & 5 & 5 \\ 2 & 6 & 4 & 6 \end{bmatrix}.
\]

Reducing to echelon form gives

\[
\mathbf{E} = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\]

Since \( \mathbf{E} \) has a row of zeroes, the vectors \( \mathbf{u}, \mathbf{v}, \mathbf{w} \) are linearly dependent.

The next theorem discusses how the columns of a matrix are affected by elementary row operations. For the following theorem, we will use the notation

\[
\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_k \end{bmatrix}
\]

to mean that \( \mathbf{A} \) is a matrix whose columns are the vectors \( \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k. \)

**Theorem 2  Row Operations and Dependence of Columns**

Consider two \( n \times k \) matrices

\[
\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_k \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k \end{bmatrix},
\]

where \( \mathbf{B} \) is obtained from \( \mathbf{A} \) using one or more elementary row operations. If the columns of \( \mathbf{A} \) satisfy an equation of the form

\[
c_1 \mathbf{a}_1 + c_2 \mathbf{a}_2 + \cdots + c_k \mathbf{a}_k = \mathbf{0},
\]

where \( c_1, c_2, \ldots, c_k \) are scalars, then the columns of \( \mathbf{B} \) satisfy the same equation:

\[
c_1 \mathbf{b}_1 + c_2 \mathbf{b}_2 + \cdots + c_k \mathbf{b}_k = \mathbf{0}.
\]

This theorem says that the relationship between the columns of a matrix does not change when we perform elementary row operations. The following example illustrates this phenomenon.
EXAMPLE 2 Consider the following matrix:

\[
\begin{bmatrix}
1 & 2 & 4 \\
3 & 0 & 6 \\
2 & 2 & 6 \\
4 & 1 & 9 \\
\end{bmatrix}
\]

The three columns of this matrix are linearly dependent. In particular, the third column is equal to twice the first column plus the second column. We can write this as an equation

\[2a_1 + a_2 - a_3 = 0,\]

where \(a_1, a_2,\) and \(a_3\) represent the columns of the matrix.

The relationship between the columns of this matrix will not change if we apply elementary row operations. For example, suppose we subtract three times the first row from the second:

\[
\begin{bmatrix}
1 & 2 & 4 \\
0 & -6 & -6 \\
2 & 2 & 6 \\
4 & 1 & 9 \\
\end{bmatrix}
\]

As you can see, the third column is still equal to twice the first column plus the second. Indeed, we can reduce this matrix all the way to reduced echelon form:

\[
\begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

It is still true that the third column is twice the first column plus the second column. Indeed, this relationship is much more obvious than it was in the original matrix.

The relationship between the columns of a reduced echelon matrix is always fairly obvious:

1. Any column with a pivot represents a vector that is independent from the previous vectors.

2. Any column without a pivot represents a vector that can be written as a linear combination of the previous vectors. In particular, the entries of the column are the coefficients of this linear combination.

Since row reduction does not change the relationship between the columns, we can row reduce a matrix to find the relationship between its column vectors.
EXAMPLE 3  Let \( \mathbf{u} = (2, 5, 3) \) and \( \mathbf{v} = (1, 6, 4) \). Express the vector \( \mathbf{w} = (7, 7, 3) \) as a linear combination of \( \mathbf{u} \) and \( \mathbf{v} \).

SOLUTION  We make a matrix whose columns are \( \mathbf{u} \), \( \mathbf{v} \), and \( \mathbf{w} \), with the desired vector \( \mathbf{w} \) in the last column:

\[
\begin{bmatrix}
2 & 1 & 7 \\
5 & 6 & 7 \\
3 & 4 & 3
\end{bmatrix}.
\]

Next, we reduce this matrix to reduced echelon form:

\[
\begin{bmatrix}
1 & 0 & 5 \\
0 & 1 & -3 \\
0 & 0 & 0
\end{bmatrix}.
\]

At this point, the relationship between the columns is clear:

\[
5 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix}.
\]

Therefore, the original columns had the same relationship:

\[
5 \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix} - 3 \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 3 \end{bmatrix}.
\]

That is, \( \mathbf{w} = 5\mathbf{u} - 3\mathbf{v} \).

Note that the third column of the reduced echelon matrix in the previous example had no pivot, since it was a linear combination of the previous columns. In general, the columns of an echelon matrix are linearly independent if and only if every column has a pivot. This motivates the following theorem.

**Theorem 3  Linear Dependence Test for Columns**

Consider an \( n \times k \) matrix

\[
\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_k \end{bmatrix}
\]

whose columns are the vectors \( \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k \). Let \( \mathbf{E} \) be an echelon matrix obtained by reducing \( \mathbf{A} \). Then \( \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k \) are linearly dependent if and only if \( \mathbf{E} \) has a column without a pivot.
EXAMPLE 4  Determine whether the three vectors $\mathbf{u} = (1, 2, 3, 2)$, $\mathbf{v} = (2, 5, 5, 5)$, and $\mathbf{w} = (2, 6, 4, 6)$ are linearly dependent.

SOLUTION  These are the same three vectors we used in Example 1. This time, we make a matrix whose columns are $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$:

$$
\mathbf{A} = \begin{bmatrix}
1 & 2 & 2 \\
2 & 5 & 6 \\
3 & 5 & 4 \\
2 & 5 & 6
\end{bmatrix}.
$$

Reducing gives

$$
\mathbf{E} = \begin{bmatrix}
1 & 0 & -2 \\
0 & 1 & 2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
$$

(Though it is only necessary to reduce to echelon form, we have reduced all the way to reduced echelon form.) The third column of $\mathbf{E}$ has no pivot, so the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are linearly dependent. Indeed, we can see that

$$
\mathbf{w} = -2\mathbf{u} + 2\mathbf{v}.
$$

Finally, the book states a test for linear dependence using determinants. Here is a version of that test:

**Theorem 4  Determinant Test**

Let $\mathbf{A}$ be an $n \times n$ square matrix. Then the following are equivalent:

1. The determinant of $\mathbf{A}$ is nonzero.
2. The matrix $\mathbf{A}$ is invertible (nonsingular).
3. The reduced echelon form for $\mathbf{A}$ is the $n \times n$ identity matrix.
4. The rows of $\mathbf{A}$ are linearly independent.
5. The columns of $\mathbf{A}$ are linearly independent.

We have already seen the equivalence of (1) and (2), and the equivalence of (2) and (3) is implicit in our row reduction algorithm for finding the inverse of a matrix. The equivalence of (3) with (4) and (5) follows from Theorem 1 and Theorem 3.

This test lets us use determinants to determine whether vectors are linearly independent:
EXAMPLE 5  Determine whether the vectors \((3, 1, 6), (2, 0, 4),\) and \((2, 1, 4)\) are linearly dependent.

SOLUTION  We compute the determinant of the matrix whose rows are the given vectors:
\[
\begin{vmatrix}
3 & 1 & 6 \\
2 & 0 & 4 \\
2 & 1 & 4 \\
\end{vmatrix} = 3(-4) - 1(0) + 6(2) = 0.
\]

Since the determinant is zero, the given vectors are linearly dependent.

In the last example, it would work just as well to make the given vectors the columns of a matrix. Also, note that this method only works if the matrix that you get is square, since you can’t take the determinant of a non-square matrix.

Theorem 4 can also be useful for recognizing when a determinant is zero:

EXAMPLE 6  Evaluate the following determinant:
\[
\begin{vmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 2 & 3 & 4 & 5 \\
2 & 3 & 4 & 5 & 6 \\
3 & 4 & 5 & 6 & 7 \\
4 & 5 & 6 & 7 & 8 \\
\end{vmatrix}
\]

SOLUTION  Observe that the third row of this matrix is equal to the sum of the first two rows. Since the rows of this matrix are linearly dependent, the determinant must be zero.