So far, the only general method we have for solving differential equations involves equations of the form
\[ y' = f(x), \]
where \( f(x) \) is any function of \( x \). The solution to such an equation is obtained by integrating:
\[ y = \int f(x) \, dx \]

In this section, we will expand upon this method by learning how to integrate both sides of an equation. This will allow us to solve many different types of differential equations.

### Implicit Differentiation

The method of integrating both sides is related to a technique from calculus known as implicit differentiation. This is the procedure we use to take the derivative (with respect to \( x \)) of an expression that involves both \( x \) and \( y \).

The idea of implicit differentiation is to imagine that \( y \) represents some unknown formula, and to simply write \( y' \) whenever we would usually write the derivative of that formula.

**EXAMPLE 1** Compute the derivative with respect to \( x \) of each of the following expressions.

(a) \( y^3 \) 
(b) \( x^2 y \) 
(c) \( x^3 e^{2y} \)

**SOLUTION**

(a) Imagine that \( y \) represents some unknown formula, such as \( \sin x \). In this case, \( y^3 \) would be \( (\sin x)^3 \), and we could take its derivative using the chain rule:
\[
\frac{d}{dx} [(\sin x)^3] = 3(\sin x)^2 \cos x.
\]

In the same way, we can use the chain rule to take the derivative of \( y^3 \):
\[
\frac{d}{dx} [y^3] = 3y^2 y'
\]

where \( y' \) represents the derivative of \( y \), whatever that may be.

(b) Again, to take the derivative of something like \( x^2 \sin x \), we would normally use the product rule:
\[
\frac{d}{dx} [x^2 \sin x] = x^2 \cos x + 2x \sin x.
\]

Therefore, we must also use the product rule to take the derivative of \( x^2 y \):
\[
\frac{d}{dx} [x^2 y] = x^2 y' + 2xy
\]

(c) Here we must use a combination of the product rule and the chain rule:
\[
\frac{d}{dx} [x^3 e^{2y}] = 2x^3 e^{2y} y' + 3x^2 e^{2y}
\]
When we are given an equation involving $x$ and $y$, we can use implicit differentiation to take the derivative of both sides.

**EXAMPLE 2** Take the derivative of both sides of each equation with respect to $x$.

(a) $y^2 + x^4 = e^{3x}$  (b) $x^3y = 4 \sin x$

**SOLUTION** The solutions are

\[
2yy' + 4x^3 = 3e^{3x} \quad \text{and} \quad x^3y' + 3x^2y = 4 \cos x
\]

**Integrating Both Sides**

The idea of integrating both sides is to reverse this process. Given a differential equation like

\[3y^2y' = 4x^3 + 2,\]

we take the antiderivative of both sides to get an equation involving $x$ and $y$:

\[y^3 = x^4 + 2x + C.\]

We can now solve for $y$ to get the general solution to the given differential equation:

\[y = \sqrt[3]{x^4 + 2x + C}.\]

**EXAMPLE 3** Find the general solution to each of the following differential equations.

(a) $y^2y' = e^{2x}$  (b) $x^3y' + 3x^2y = 1$  (c) $2yy' \ln x + \frac{y^2}{x} = 0$

**SOLUTION**

(a) We can take the antiderivative of both sides to get

\[\frac{1}{3}y^3 = \frac{1}{2}e^{2x} + C.\]

Solving for $y$ yields $y = \sqrt[3]{\frac{3}{2}e^{2x} + 3C}$, or equivalently $y = \sqrt[3]{\frac{3}{2}e^{2x} + C}$.

(b) The left side of this equation looks like a result of the product rule. Therefore, the antiderivative is

\[x^3y = x + C.\]

Solving for $y$ gives $y = \frac{x^{-2} + Cx^{-3}}{3}$.

(c) Again, the left side is a result of the product rule, with the two factors being $y^2$ and $\ln x$. Therefore, the antiderivative is

\[y^2 \ln x = C.\]

Solving for $y$ gives $y = \pm \sqrt[3]{\frac{C}{\ln x}}$, or equivalently $y = \frac{C}{\sqrt{\ln x}}$.

Sometimes it necessary to manipulate an equation algebraically before it can be integrated.
Implicit Solutions

Sometimes after you integrate both sides of a differential equation it is not possible to algebraically solve for $y$. For example, consider the equation

$$3y^2 y' + e^y y' = x^2.$$  

Integrating both sides gives

$$y^3 + e^y = \frac{1}{3}x^3 + C.$$  

Unfortunately, there is no way to solve this equation algebraically for $C$. This implicit solution still describes $y$ as a function of $x$, in the sense that the curves in the $xy$-plane defined by this equation are graphs of the solutions. However, there is no way to write a formula for these solutions algebraically.

**EXAMPLE 4** Find the general solution to each of the following differential equations.

(a) $y' = \frac{\cos x}{4y^3}$  
(b) $x^3 y' = x^5 - 3x^2 y$

**SOLUTION**

(a) The right side of this equation cannot be integrated, since it has $y$ but not $y'$. However, we can make this equation integrable if we multiply through by $4y^3$:

$$4y^3 y' = \cos x.$$  

Integrating both sides gives

$$y^4 = \sin x + C$$  

so the general solution is $y = \pm \sqrt[4]{\sin x + C}$.

(b) This time, neither side of the equation can be integrated. However, we can make this equation integrable if we move the $3x^2 y$ to the left side:

$$x^3 y' + 3x^2 y = x^5.$$  

The left side is now a result of the product rule. Integrating both sides gives

$$x^3 y = \frac{1}{6}x^6 + C$$  

so the general solution is $y = \frac{1}{6}x^3 + Cx^{-3}$.

Separation of Variables

For many equations, the best way to make the equation integrable is to separate the variables so that all the $x$’s lie on one side of the equation, and all of the $y$’s lie on the other side. For example, consider the equation

$$-y' = 3x^2 e^y.$$  

To make this equation integrable, the best strategy is to divide through by $e^y$, separating the two variables:

$$-e^{-y} y' = 3x^2$$
Integrating Second-Order Equations

The method of integrating both sides can also be used on second-order equations, although the antiderivatives are often much more difficult. For example, consider the equation

\[ yy'' + (y')^2 = 6x. \]

Though it may not be obvious, the left side is a result of the product rule: it is the derivative of the product \( yy' \). Integrating both sides gives

\[ yy' = 3x^2 + C_1. \]

We have now reduced to a first-order equation, and integrating both sides again yields

\[ \frac{1}{2}y^2 = x^3 + C_1x + C_2. \]

Therefore, the general solution is

\[ y = \pm \sqrt{2x^3 + C_1x + C_2}. \]

Integrating both sides yields

\[ e^{-y} = x^2 + C \]

and solving for \( y \) gives \( y = -\ln(x^2 + C) \).

In general, if you can put an equation into the form

\[ f(y)y' = g(x), \]

then you can integrate both sides by computing a pair of integrals.

### Separation of Variables

Given any equation of the form

\[ f(y)y' = g(x), \]

we can integrate both sides to get

\[ F(y) = G(x) + C \]

where \( F(y) = \int f(y) \, dy \), and \( G(x) = \int g(x) \, dx \).

#### Example 5

Find the general solution to each of the following differential equations.

(a) \( y'e^x = xy^2 \)  
(b) \( xy' = \cos^2 y \)

**Solution**

(a) To separate the variables, we must divide through by both \( e^x \) and \( y^2 \). This gives

\[ \frac{1}{y^2}y' = xe^{-x} \]

We can now integrate both sides, using integration by parts for \( \int xe^{-x} \, dx \). The result is

\[ -y^{-1} = -(x + 1)e^{-x} + C. \]

Solving for \( y \) gives \( y = \frac{1}{(x + 1)e^{-x} - C} \) or equivalently \( y = \frac{1}{(x + 1)e^{-x} + C} \).
Separation of Variables using Differentials

There is a nice way of viewing separation of variables using differentials, which looks a little bit different from our method. Given an equation like
\[ e^x y' = x y^3, \]
we can write it as
\[ e^x dy = x y^3 dx, \]
and then “separate variables” as follows:
\[ y^{-3} dy = x e^{-x} dx. \]

Here, the \( dx \) and \( dy \) by themselves are “differentials”, which represent small changes in the values of \( x \) and \( y \). Then
\[ \int y^{-3} dy = \int x e^{-x} dx, \]
and evaluating these two integrals gives the solution.

(b) To separate variables, we must divide through by both \( x \) and \( \cos^2 y \). This gives
\[ (\sec^2 y) y' = \frac{1}{x} \]

We can now integrate both sides, which yields
\[ \tan y = \ln |x| + C, \]
so the general solution is
\[ y = \tan^{-1} (\ln |x| + C). \]

EXAMPLE 6  Find the general solution to the equation
\[ y' = y \cos x. \]

SOLUTION  We can divide through by \( y \) to separate variables:
\[ \frac{y'}{y} = \cos x. \]
Integrating both sides yields
\[ \ln |y| = \sin x + C. \]
To solve for \( y \), we must take the exponential of both sides. This gives
\[ |y| = e^{\sin x + C}, \]
which is the same as
\[ y = \pm e^{\sin x + C}. \]

There is a complicated way to simplify this formula. First, we split up the exponential:
\[ y = \pm e^C e^{\sin x}. \]

Now, since \( C \) is an arbitrary constant, \( e^C \) may be any positive constant. Then \( \pm e^C \) may be positive or negative, so it is essentially an arbitrary constant. Therefore, we can write the general solution as
\[ y = Ce^{\sin x}. \]
The trick used at the end of this example is fairly common. Given a general solution of the form
\[ |y| = e^{f(x) + C} \]
we can use this trick to rewrite the solution as
\[ y = Ce^{f(x)}. \]

**Integrating Factors**

Consider the following differential equation:
\[ xy' + 3y = x^4. \]

This equation cannot be integrated directly, and it is not obvious how to perform algebra to make it integrable. In particular, it certainly won’t work to separate the variables.

There is actually a very clever trick we can use to make this equation integrable. We need to *multiply through by* \( x^2 \):
\[ x^3y' + 3x^2y = x^6. \]

As you can see, the left side is now a result of the product rule. Integrating both sides gives
\[ x^3y = \frac{1}{7}x^7 + C \]
so \( y = \frac{1}{7}x^4 +Cx^{-3} \).

The factor of \( x^2 \) that we multiplied by is called an **integrating factor**—a factor that you multiply to make an equation integrable. There is a well-known class of differential equations for which integrating factors can be quite helpful.

**FIRST-ORDER LINEAR EQUATIONS**

A differential equation of the form
\[ f(x)y'' + g(x)y' + h(x)y = k(x). \]

is called a **second-order linear equation**. We will be discussing such equations later in the course.

It turns out that any **first-order linear equation can be made integrable if you multiply by the appropriate integrating factor**. When searching for this factor, keep in mind that the goal is to make the derivative of \( f(x) \) equal to \( g(x) \).

**EXAMPLE 7** Find the general solution to each of the following differential equations.

(a) \[ x^2y' + 6xy = x + 1 \]
(b) \[ y' + 2y = x \]
(c) \[ y' - y \tan x = 1 \]

**SOLUTION** All of these equations are linear, so we ought to be able to find integrating factors.

(a) This equation isn’t integrable, since \( 6x \) isn’t the derivative of \( x^2 \). However, we can make it integrable if we multiply through by \( x^4 \):
\[ x^6y' + 6x^5y = x^5 + x^4. \]
The Integrating Factor Formula

There is actually a formula for the integrating factor required to make a first-order linear equation integrable. Given the equation

\[ y' + g(x)y = h(x) \]

the appropriate integrating factor is \( e^{\int g(x) \, dx} \), where \( G \) is any antiderivative of \( g \). Note that any first-order linear equation can be brought into this form by dividing through by the coefficient of \( y' \).

Although this formula works for any first-order linear equation, it can be hard to remember, and the algebra is often unnecessarily complicated. For simple equations, it is usually much easier to guess the integrating factor and then work from there.

The left side is now the derivative of \( x^6y \). Integrating both sides gives

\[ x^6y = \frac{1}{6}x^6 + \frac{1}{5}x^5 + C \]

Solving for \( y \) gives

\[ y = \frac{1}{6} + \frac{1}{5}x^{-1} + Cx^{-6} \]

(b) Multiplying by a power of \( x \) won’t work here, since \( 2x^a \) is never the derivative of \( x^a \). Instead, we need to multiply through by \( e^{2x} \):

\[ e^{2x}y' + 2e^{2x}y = xe^{2x} \]

The left side is now the derivative of \( e^{2x}y \). We can now integrate both sides, using integration by parts on the right side.

\[ e^{2x}y = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C \]

Solving for \( y \) gives

\[ y = \frac{1}{2}x - \frac{1}{4} + Ce^{-2x} \]

(c) A little bit of guessing and checking is required to find an integrating factor for this equation. The right strategy is to multiply through by \( \cos x \):

\[ y' \cos x - y \sin x = \cos x \]

The left side is the derivative of \( y \cos x \). Integrating both sides gives

\[ y \cos x = \sin x + C \]

so \( y = \tan x + C \sec x \).
EXERCISES

1–6 ■ Find the general solution to the given differential equation by integrating both sides. You may need to use some algebra to put the equation into integrable form.

1. $2yy' = e^x$
2. $y' \cos y = x^3$
3. $x^4y' + 4x^3y = x^4$
4. $e^{xy} y' + 3e^{xy} y = e^{5x}$
5. $x^2y' = x - 2xy$
6. $(y + y')e^x = \sec^2 x$

7–14 ■ Use separation of variables to find the general solution to the given differential equation.

7. $y' = x \sec y$
8. $e^{-x}y' = y^2$
9. $e^{x+y}y' = 1$
10. $(x^2 + 1)y' = e^{-y}$
11. $x^2y' = 2xy^2 + y'$
12. $y' - y^2 = 1 - x^2y'$
13. $y' \cos^2 x = y$
14. $xy' = 2y$
15–22 ■ Use an integrating factor to find the general solution to the given differential equation.

15. $x^2y' + 3xy = 1$
16. $xy' - 2y = x^5$
17. $y' + 5y = e^x$
18. $y' - y = x$
19. $y' + y \cot x = \cos x$
20. $y' \sin x + y \sec x = \cos x$
21. $xy' \ln x + y = x^2$
22. $y' + 2xy = x$
23–24 ■ Solve the given initial value problem.

23. $y' = -3x^2y^2, \ y(0) = 3$
24. $y' = x/y, \ y(3) = -5$
25. $xy' + 2y = x^2, \ y(1) = 1$
26. $y' = e^x + 3y, \ y(0) = 4$
Answers

1. $y = \pm \sqrt{e^x + C}$  
2. $y = \sin^{-1} \left( \frac{x^4}{4} + C \right)$  
3. $y = \frac{x}{5} + \frac{C}{x^4}$  
4. $y = \frac{1}{5} e^{2x} + Ce^{-3x}$  
5. $y = \frac{1}{2} + Cx^{-2}$

6. $y = e^{-x}(\tan x + C)$  
7. $y = \sin^{-1} \left( \frac{1}{2} x^2 + C \right)$  
8. $y = \frac{1}{C - e^x}$  
9. $y = \ln \left( c - e^{-x} \right)$  
10. $y = \ln \left( \arctan x + C \right)$

11. $y = \frac{1}{C - \ln(x^2 - 1)}$  
12. $y = \tan \left( \arctan x + C \right)$  
13. $y = Ce^{\tan x}$  
14. $y = Cx^2$  
15. $y = \frac{1}{2x} + \frac{C}{x^3}$

16. $y = \frac{x^5}{3} + Cx^2$  
17. $y = \frac{e^x}{6} + Ce^{-5x}$  
18. $y = Ce^x - x - 1.$

19. $y = \frac{1}{2} \sin x + C \csc x$ or $y = -\frac{1}{2} \cos x \cot x + C \csc x.$  
20. $y = (x + C) \cot x.$  
21. $y = \frac{x^2 + C}{2 \ln x}$

22. $y = \frac{1}{2} + Ce^{-x^2}$  
23. $y = \frac{3}{3x^3 + 1}$  
24. $y = -\sqrt{x^2 + 16}$  
25. $y = \frac{x^4 + 3}{4x^2}$  
26. $y = \frac{9e^{3x} - e^x}{2}$