2. Limits at Infinity

To understand sequences and series fully, we will need to have a better understanding of limits at infinity. We begin with a few examples to motivate our discussion.

**EXAMPLE 1** Find \( \lim_{n \to \infty} \frac{2n + 4}{n + 1} \).

**SOLUTION** Both the numerator and denominator of the fraction are approaching infinity. Unfortunately, this doesn't tell us anything about the limit—it depends on how the two infinities compare. In the language of calculus, \( \infty / \infty \) is an indeterminate form.

Let's try plugging in some values for \( n \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>2n + 4</th>
<th>n + 1</th>
<th>( \frac{2n + 4}{n + 1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>24</td>
<td>11</td>
<td>2.1818</td>
</tr>
<tr>
<td>100</td>
<td>204</td>
<td>101</td>
<td>2.0198</td>
</tr>
<tr>
<td>1000</td>
<td>2,004</td>
<td>1,001</td>
<td>2.0020</td>
</tr>
<tr>
<td>10,000</td>
<td>20,004</td>
<td>10,001</td>
<td>2.0002</td>
</tr>
</tbody>
</table>

For large values of \( n \), the numerator is almost exactly twice the denominator, so the ratio approaches 2.

What's happening is that the constant terms (+4 in the numerator and +1 in the denominator) are becoming less and less important as \( n \) grows larger. As a result, these constants play no role in the outcome:

\[
\lim_{n \to \infty} \frac{2n + 4}{n + 1} = \lim_{n \to \infty} \frac{2n}{n} = 2.
\]

As a general rule, finding a limit as \( n \to \infty \) often has to do with figuring out which parts of a formula you can ignore. The trick is to concentrate on the largest term of any sum, ignoring the smaller terms. For a polynomial, this dominant term is whichever term has the highest degree, i.e. the highest power of \( n \).
**THEOREM (LIMITS OF RATIONAL FUNCTIONS)**

Let \( p(n) \) and \( q(n) \) be nonzero polynomials, and let \( an^j \) and \( bn^k \) be their respective dominant terms. Then

\[
\lim_{n \to \infty} \frac{p(n)}{q(n)} = \lim_{n \to \infty} \frac{an^j}{bn^k}.
\]

**EXAMPLE 2**  Evaluate the following limits.

(a) \( \lim_{n \to \infty} \frac{n^2 + 5n}{1 + 4n^3} \)

(b) \( \lim_{n \to \infty} \frac{n^4 + 1}{6n^3 + 2n + 5} \)

(c) \( \lim_{n \to \infty} \frac{1 + 5n - 6n^2}{3n^2 + 4} \)

**SOLUTION**  We use the above theorem in each case.

(a) The dominant term in the numerator is the \( n^2 \), and the dominant term in the denominator is the \( 4n^3 \). Thus

\[
\lim_{n \to \infty} \frac{n^2 + 5n}{1 + 4n^3} = \lim_{n \to \infty} \frac{n^2}{4n^3} = \lim_{n \to \infty} \frac{1}{4n} = 0.
\]

(c) The dominant term in the numerator is the \( n^4 \), and the dominant term in the denominator is the \( 6n^3 \). Thus

\[
\lim_{n \to \infty} \frac{n^4 + 1}{6n^3 + 2n + 5} = \lim_{n \to \infty} \frac{n^4}{6n^3} = \lim_{n \to \infty} \frac{n}{6} = \infty.
\]

(c) The dominant term in the numerator is the \( -6n^2 \) (including the negative sign). The dominant term in the denominator is the \( 3n^2 \), and thus

\[
\lim_{n \to \infty} \frac{1 + 5n - 6n^2}{3n^2 + 4} = \lim_{n \to \infty} \frac{-6n^2}{3n^2} = -2. \quad \blacksquare
\]

Our goal is to expand upon this method by evaluating the sizes of different quantities as \( n \to \infty \). The ideas discussed here are the beginning of a field of math known as **asymptotic analysis**, and are used throughout mathematics and science (especially physics and computer science) to evaluate the rate at which quantities grow or shrink as \( n \to \infty \).
The Asymptotic Hierarchy

We begin by defining precisely what it means for one quantity to be much smaller than another.

**DEFINITION OF $\ll$**

Let $\{a_n\}$ and $\{b_n\}$ be sequences. We say that $a_n$ is much smaller than $b_n$ as $n \to \infty$, and write

$$a_n \ll b_n \text{ as } n \to \infty$$

if

$$\lim_{n \to \infty} \frac{a_n}{b_n} = 0.$$

The symbol $\ll$, which means “is much smaller than”, consists of two less than symbols (\(<\)) in a row. We will also use the symbol $\gg$ for “is much larger than”, where $a_n \gg b_n$ means the same thing as $b_n \ll a_n$.

**EXAMPLE 3** For a basic example, observe that $n$ is much smaller than $n^2$ as $n \to \infty$, written

$$n \ll n^2 \text{ as } n \to \infty.$$

This is because

$$\lim_{n \to \infty} \frac{n}{n^2} = \lim_{n \to \infty} \frac{1}{n} = 0.$$

Intuitively, the statement “$n^2 \gg n$ as $n \to \infty$” expresses the idea that “infinity squared is much larger than infinity”.

More generally, we have the following rule.

**THEOREM (COMPARING POWERS)**

If $p < q$, then

$$n^p \ll n^q \text{ as } n \to \infty.$$  

This rule even works in the case where $p$ and $q$ are non-integers. For example,

$$\sqrt[3]{n} \ll \sqrt{\sqrt{n}} \ll n \quad \text{as } n \to \infty$$

since $1/3 < 1/2 < 1$. 
EXAMPLE 4  How do polynomials compare with exponentials? For example, do you think that $n^5$ or $2^n$ is larger as $n \to \infty$?

The following table compares these sequences for different values of $n$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$n^5$</th>
<th>$2^n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>3,125</td>
<td>32</td>
</tr>
<tr>
<td>10</td>
<td>100,000</td>
<td>1,024</td>
</tr>
<tr>
<td>15</td>
<td>759,375</td>
<td>32,768</td>
</tr>
<tr>
<td>20</td>
<td>800,000</td>
<td>1,048,576</td>
</tr>
<tr>
<td>50</td>
<td>312,500,000</td>
<td>1,125,899,906,842,624</td>
</tr>
<tr>
<td>100</td>
<td>10,000,000,000</td>
<td>1,267,650,600,228,229,401,496,703,205,376</td>
</tr>
</tbody>
</table>

As you can see, $2^n$ grows much more quickly than $n^5$, so $n^5 \ll 2^n$ as $n \to \infty$.

In the last example, it may surprise you that $2^{100}$ is so much larger than $100^5$. The idea is that multiplication is very powerful: $2^{100}$ is larger because it is the product of 100 different things, while $100^5$ is the product of only five things. This reasoning applies whenever you are comparing an exponential with a polynomial.

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THEOREM (EXPONENTS VS. POWERS)
For any value of $p$ and any $r > 1$, we have

$$n^p \ll r^n \text{ as } n \to \infty.$$  

For example, it turns out that $n^{100}$ is much smaller than $(1.01)^n$ as $n \to \infty$. Though $(1.01)^n$ gets off to a slow start, it overtakes $n^{100}$ starting at $n = 117,308$, and grows much more quickly than $n^{100}$ from that point on.

Incidentally, the rule for comparing exponentials is very simple.

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THEOREM (COMPARING EXPONENTIALS)
If $0 \leq r < s$, then

$$r^n \ll s^n \text{ as } n \to \infty.$$
EXAMPLE 5  How do logarithms compare with polynomials and exponentials?  If you think about it, log $n$ actually grows very slowly as $n \to \infty$. For example, here is a table comparing $\sqrt{n}$ and log $n$ for different values of $n$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>log $n$</th>
<th>$\sqrt{n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>10,000</td>
<td>4</td>
<td>100</td>
</tr>
<tr>
<td>1,000,000</td>
<td>6</td>
<td>1000</td>
</tr>
<tr>
<td>100,000,000</td>
<td>8</td>
<td>10,000</td>
</tr>
</tbody>
</table>

As you can see, log $n \ll \sqrt{n}$ as $n \to \infty$.

It turns out that log $n$ grows more slowly than any power of $n$. For example, log $n$ is much smaller than $n^{0.01}$ (the hundredth root of $n$) for large values of $n$.

**THEOREM (LOGARITHMS VS. POWERS)**
For any $p > 0$, we have

$$\log n \ll n^p \text{ as } n \to \infty.$$ 

Note that logarithms with different bases are simply multiples of one another. For example,

$$\ln n = \frac{\log n}{\log e} \approx 2.3 \log n.$$ 

Thus, the base of a logarithm doesn't affect its size very much.

EXAMPLE 6  So far, the largest quantities we have found are the exponentials:

$$2^n \ll 3^n \ll 4^n \ll \ldots.$$ 

Is anything larger than all of these?

Yes. Recall that the factorial of $n$ (written $n!$) is the product of all numbers from 1 to $n$:

$$n! = 1 \times 2 \times 3 \times \cdots \times n.$$ 

It is not hard to see that $n!$ must be larger than $2^n$. For example, here is a comparison of $2^{100}$ and 100 factorial:

$$2^{100} = 2 \times 2 \times 2 \times \cdots \times 2 \times 2$$

$$100! = 1 \times 2 \times 3 \times \cdots \times 99 \times 100$$
Both are the product of 100 factors, but in the case of 100! most of the factors are much larger than 2. For this same reason, \( n! \) will be larger than \( r^n \) for any base \( r \).

Much larger than even \( n! \) is the quantity \( n^n \). For example, \( 100^{100} \) is clearly much larger than 100!.

\[
100! = 1 \times 2 \times 3 \times \cdots \times 99 \times 100 \\
100^{100} = 100 \times 100 \times 100 \times \cdots \times 100 \times 100 
\]

For comparison, \( 2^{100} \) is a 31-digit number, 100! is a 158-digit number, and \( 100^{100} \) is a 201-digit number.

We now have a fairly clear picture of the hierarchy of functions.

---

THE ASYMPTOTIC HIERARCHY

\[
\ln n \ll \cdots \ll \sqrt{n} \ll n \ll n^2 \ll \cdots \ll 2^n \ll 3^n \ll \cdots \ll n! \ll n^n
\]

Knowing the positions of functions in this hierarchy can make certain limits very easy to evaluate.

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THEOREM (LIMITS OF RATIOS)

Let \( \{a_n\} \) and \( \{b_n\} \) be sequences.

(a) If \( a_n \ll b_n \) as \( n \to \infty \), then \( \lim_{n \to \infty} \frac{a_n}{b_n} = 0 \).

(b) If \( a_n \) and \( b_n \) are positive and \( a_n \gg b_n \) as \( n \to \infty \), then \( \lim_{n \to \infty} \frac{a_n}{b_n} = \infty \).

---

EXAMPLE 7  Evaluate the following limits.

(a) \( \lim_{n \to \infty} \frac{\ln n}{n^3} \)  \qquad (b) \( \lim_{n \to \infty} \frac{5^n}{n!} \)  \qquad (c) \( \lim_{n \to \infty} \frac{2^n}{n^3} \)

SOLUTION  Limit (a) is 0 since \( \ln n \ll n^3 \) as \( n \to \infty \). Limit (b) is also 0 since \( 5^n \ll n! \) as \( n \to \infty \). Finally, limit (c) is \( \infty \) since \( 2^n \) and \( n^3 \) are both positive and \( 2^n \gg n^3 \) as \( n \to \infty \).

The following theorem lists a few more important properties of \( \ll \).
THEOREM (PROPERTIES OF $\ll$)

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences, and let $C$ and $D$ be nonzero constants.

1. If $a_n \ll b_n$ and $b_n \ll c_n$ as $n \to \infty$, then $a_n \ll c_n$ as $n \to \infty$.

2. If $a_n \ll b_n$ as $n \to \infty$, then $Ca_n \ll Db_n$ as $n \to \infty$.

Part (2) of this theorem essentially says that you can ignore coefficients when comparing the sizes of expressions. For example,

$$3 \log n \ll 5 \sqrt{n} \text{ as } n \to \infty$$

since $\log n \ll \sqrt{n}$.

We are now ready to define a more general notion of dominant terms.

DEFINITION (DOMINANT TERMS)

A term of a sum is dominant if it is much larger than every other term.

EXAMPLE 8

Find the dominant term in each of the following sums.

(a) $8n^4 + 5n^3 + 2^n$  
(b) $\sqrt{n} + 5(n!) + 10^n$  
(c) $n \sqrt{n} + \sqrt[3]{n} + \log n$

SOLUTION

The dominant term in (a) is the $2^n$. The dominant term in (b) is the $5(n!)$. The dominant term in (c) is the $n \sqrt{n}$. (Note that $n \sqrt{n} \gg \sqrt[3]{n}$, since $3/2 > 1/5$.)

Asymptotic Equivalence

In many cases, it is only necessary to consider the dominant term in a sum. For example, in a quotient of two polynomials, we can replace each polynomial by its dominant term when computing the limit:

$$\lim_{n \to \infty} \frac{n^3 + 7}{6n^3 + 2n - 1} = \lim_{n \to \infty} \frac{n^3}{6n^3} = \frac{1}{6}.$$ 

The idea is that the original numerator is roughly the same as just $n^3$, since the $+7$ doesn't contribute very much. Similarly, the original denominator is roughly the same as just $6n^3$, since the other two terms are much smaller.

The following definition makes precise the idea that two quantities are “roughly the same”.
DEFINITION (ASYMPTOTIC EQUIVALENCE)
Two sequences \( \{a_n\} \) and \( \{b_n\} \) are \textbf{asymptotically equivalent}, written
\[
a_n \sim b_n \quad \text{as} \quad n \to \infty
\]
if
\[
\lim_{n \to \infty} \frac{a_n}{b_n} = 1.
\]

For example, \( n^3 + 7 \) is asymptotically equivalent to \( n^3 \), since
\[
\lim_{n \to \infty} \frac{n^3 + 7}{n^3} = 1.
\]

More generally:
\begin{quote}
Any sum is asymptotically equivalent to its dominant term.
\end{quote}

For example, \( 4n^5 + \sqrt{n} + 3^9 \) is asymptotically equivalent to \( 3^9 \), since
\[
\lim_{n \to \infty} \frac{4n^5 + \sqrt{n} + 3^9}{3^9} = 1.
\]

The following theorem lets us use asymptotic equivalence to compute limits.

THEOREM (EVALUATING LIMITS)
1. Asymptotically equivalent sequences have the same limit. That is, if \( a_n \sim A_n \) as \( n \to \infty \), then
\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} A_n.
\]
2. If \( a_n \sim A_n \) and \( b_n \sim B_n \) as \( n \to \infty \), then
\[
\frac{a_n}{b_n} \sim \frac{A_n}{B_n}.
\]

EXAMPLE 9  To compute the limit
\[
\lim_{n \to \infty} \frac{n^3 + 7\sqrt{n}}{5n^3 + \ln n},
\]
observe that \( n^3 + 7\sqrt{n} \sim n^3 \) and \( 5n^3 + \ln n \sim 5n^3 \). Using part (2) of the previous theorem,
we conclude that
\[
\frac{n^3 + 7\sqrt{n}}{5n^3 + \ln n} \sim \frac{n^3}{5n^3}.
\]
and therefore, using part (1) of the previous theorem,
\[
\lim_{n \to \infty} \frac{n^3 + 7\sqrt{n}}{5n^3 + \ln n} = \lim_{n \to \infty} \frac{n^3}{5n^3} = \frac{1}{5}.
\]

Of course, you don’t need to give this much detail when you’re doing calculations. Just keep in mind that replacing the numerator or denominator of a fraction with its dominant term is safe.

**EXAMPLE 10** Find \(\lim_{n \to \infty} \frac{3n^2 + 7n + 2^n + \ln n}{5n^2 + \sqrt{n} + n!} \).

**SOLUTION** The \(2^n\) is the dominant term in the numerator, and the \(n!\) is the dominant term in the denominator. Therefore,
\[
\lim_{n \to \infty} \frac{3n^2 + 7n + 2^n + \ln n}{5n^2 + \sqrt{n} + n!} = \lim_{n \to \infty} \frac{2^n}{n!} = 0.
\]

**Further Techniques**

There are a few more cases where asymptotic techniques may be used to evaluate limits.

**THEOREM (ANALYZING POWERS)**

Let \(\{a_n\}\) and \(\{A_n\}\) be sequences. If \(a_n \sim A_n\) as \(n \to \infty\), then
\[
(a_n)^p \sim (A_n)^p \text{ as } n \to \infty
\]
for any constant power \(p\). In particular,
\[
\sqrt[n]{a_n} \sim \sqrt[n]{A_n} \text{ as } n \to \infty.
\]
EXAMPLE 11 Evaluate \( \lim_{n \to \infty} \frac{\sqrt{n^2 + 6n + 4}}{3n + 1} \).

SOLUTION Since \( n^2 + 6n + 4 \sim n^2 \), it follows that
\[ \sqrt{n^2 + 6n + 4} \sim \sqrt{n^2} = n. \]

Then
\[ \lim_{n \to \infty} \frac{\sqrt{n^2 + 6n + 4}}{3n + 1} = \lim_{n \to \infty} \frac{n}{3n} = \frac{1}{3}. \]

EXAMPLE 12 Evaluate \( \lim_{n \to \infty} \frac{(2n^2 + 1)^3}{n^6 + 4n} \).

SOLUTION Since \( 2n^2 + 1 \sim 2n^2 \), it follows that
\[ (2n^2 + 1)^3 \sim (2n^2)^3 = 8n^6. \]

Then
\[ \lim_{n \to \infty} \frac{(2n^2 + 1)^3}{n^6 + 4n} = \lim_{n \to \infty} \frac{8n^6}{n^6} = 8. \]

THEOREM (ANALYZING SIZES)
Let \( \{a_n\}, \{b_n\}, \{A_n\}, \) and \( \{B_n\} \) be sequences, where \( a_n \sim A_n \) and \( b_n \sim B_n \) as \( n \to \infty \). If
\[ A_n \ll B_n \text{ as } n \to \infty \]
then
\[ a_n \ll b_n \text{ as } n \to \infty \]
as well.

This theorem says that we can use asymptotic equivalence when analyzing sizes of terms. For example, we can use this theorem to conclude that
\[ \sqrt{n^2 + 3} \ll (n + 2 \ln n)^3. \]
The reason is that the left side is asymptotically equivalent to \( n \), the right side is asymptotically equivalent to \( n^3 \), and \( n \ll n^3 \).
EXAMPLE 13  
Find \( \lim_{n \to \infty} \frac{5n^4 + (2n^2 + 1)^3}{\sqrt{n^{12} + 2}} \).

SOLUTION  
Observe that \((2n^2 + 1)^3 \sim (2n^2)^3 = 8n^6\), which means that \((2n^2 + 1)^3\) is the dominant term in the numerator. Then

\[
\lim_{n \to \infty} \frac{5n^4 + (2n^2 + 1)^3}{\sqrt{n^{12} + 2}} = \lim_{n \to \infty} \frac{(2n^2 + 1)^3}{\sqrt{n^{12} + 2}} = \lim_{n \to \infty} \frac{(2n^2)^3}{\sqrt{n^{12}}} = \lim_{n \to \infty} \frac{8n^6}{n^6} = 8.
\]

THEOREM (ANALYZING PRODUCTS)  
Let \(\{a_n\}, \{b_n\}, \{A_n\}, \) and \(\{B_n\}\) be sequences, where \(a_n \sim A_n\) and \(b_n \sim B_n\) as \(n \to \infty\). Then

\[a_nb_n \sim A_nB_n\ \text{as}\ n \to \infty.\]

EXAMPLE 14  
Find \( \lim_{n \to \infty} \frac{(2^n + 5)(3^n + \sqrt{n})}{4n^2 + 6^n} \).

SOLUTION  
Observe that \(2^n + 5 \sim 2^n\) and \(3^n + \sqrt{n} \sim 3^n\). It follows that

\[(2^n + 5)(3^n + \sqrt{n}) \sim (2^n)(3^n) = 6^n,
\]

so

\[
\lim_{n \to \infty} \frac{(2^n + 5)(3^n + \sqrt{n})}{4n^2 + 6^n} = \lim_{n \to \infty} \frac{6^n}{6^n} = 1.
\]

THEOREM (ANALYZING LOGARITHMS)  
Let \( \{a_n\} \) and \( \{A_n\} \) be sequences, where \(a_n \sim A_n\) as \(n \to \infty\), and suppose that

\[\lim_{n \to \infty} a_n = \infty.\]

Then

\[\ln(a_n) \sim \ln(A_n)\ \text{as}\ n \to \infty.\]

EXAMPLE 15  
Find \( \lim_{n \to \infty} \frac{n + \sqrt{n}}{\ln(2^n + n^3 + 1)} \).

SOLUTION  
Since \(2^n + n^3 + 1 \sim 2^n\), it follows that

\[\ln(2^n + n^3 + 1) \sim \ln(2^n) = n \ln 2.\]
Then

\[
\lim_{n \to \infty} \frac{n + \sqrt{n}}{\ln(2^n + n^3 + 1)} = \lim_{n \to \infty} \frac{n}{n \ln 2} = \frac{1}{\ln 2}.
\]

**Pitfalls**

There are a few important pitfalls to avoid when reasoning asymptotically.

**EXAMPLE 16** Consider the limit

\[
\lim_{n \to \infty} \sqrt{n^2 + 4n} - n.
\]

It might seem that the 4\(n\) under the first square root isn't contributing much, so that

\[
\lim_{n \to \infty} \sqrt{n^2 + 4n} - n \approx \lim_{n \to \infty} \sqrt{n^2} - n = \lim_{n \to \infty} n - n = 0.
\]

But this is not correct: the actual value of the limit is 2, as illustrated in the following table.

<table>
<thead>
<tr>
<th>(n)</th>
<th>(\sqrt{n^2 + 4n})</th>
<th>(\sqrt{n^2 + 4n} - n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>101.98039</td>
<td>1.98039</td>
</tr>
<tr>
<td>1000</td>
<td>1001.99800</td>
<td>1.99800</td>
</tr>
<tr>
<td>10,000</td>
<td>10,001.99980</td>
<td>1.99980</td>
</tr>
<tr>
<td>100,000</td>
<td>100,001.99998</td>
<td>1.99998</td>
</tr>
</tbody>
</table>

The problem here is that **you can't ignore smaller terms in a sum if the big terms cancel**. In this case, what's happening is that \(n^2 + 4n\) is only approximately \(n^2\), which means that \(\sqrt{n^2 + 4n}\) is only approximately \(n\). Indeed,

\[
\sqrt{n^2 + 4n} = n + \begin{pmatrix} \text{some smaller} \\
\text{stuff} \end{pmatrix}
\]

When we subtract \(n\), what's left over isn't zero — it's the smaller stuff! In this case, the smaller stuff is getting closer and closer to 2.

By the way, there **is** a nice way to evaluate this limit algebraically. The strategy is to multiply the numerator and denominator by the conjugate of the radical expression:
\[
\lim_{n \to \infty} \sqrt{n^2 + 4n} - n = \lim_{n \to \infty} \frac{(\sqrt{n^2 + 4n} - n)(\sqrt{n^2 + 4n} + n)}{\sqrt{n^2 + 4n} + n} \\
= \lim_{n \to \infty} \frac{n^2 + 4n - n^2}{\sqrt{n^2 + 4n} + n} \\
= \lim_{n \to \infty} \frac{4n}{\sqrt{n^2 + 4n} + n}
\]

Since \(\sqrt{n^2 + 4n} \sim n\), the denominator is asymptotically equivalent to \(2n\), and therefore the limit is 2.

**EXAMPLE 17** Consider the limit

\[
\lim_{n \to \infty} \frac{2^{n+1}}{2^n + 1}.
\]

Clearly the denominator is asymptotically equivalent to \(2^n\). What about the numerator? You might be tempted to change the \(n + 1\) to an \(n\) but this is not correct! The reason is that **the exponent in an exponential matters a lot**. Small changes in an exponent can affect the size of an expression a lot. In general, if \(a_n \sim A_n\) and \(r\) is a constant, it does **not** follow that \(r^{a_n} \sim r^{A_n}\). The key to evaluating this limit is to leave the \(n + 1\) as it is:

\[
\lim_{n \to \infty} \frac{2^{n+1}}{2^n + 1} = \lim_{n \to \infty} \frac{2^{n+1}}{2^n} = 2.
\]

By the way, the same warning holds for factorials: just because \(a_n \sim A_n\) does not mean that \((a_n)! \sim (A_n)!\). As with exponents, **the inside of a factorial expression matters a lot**, and it is not safe to ignore small terms.

**Hardy vs. Landau**

The \(\ll\) and \(\sim\) notation we have been using is known as **Hardy notation**, named after British mathematician G. H. Hardy. It is popular in some branches of mathematics as well as physics.

As it happens, there is a competing notation for the concepts known as **Landau notation**, named for German mathematician Edmund Landau. The Landau notation is popular in certain other branches of mathematics as well as computer science. The following table compares the two notations.
<table>
<thead>
<tr>
<th>Hardy Notation</th>
<th>Landau Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_n \ll b_n$</td>
<td>$a_n = o(b_n)$</td>
</tr>
<tr>
<td>$a_n \sim Cb_n$</td>
<td>$a_n = \Theta(b_n)$</td>
</tr>
</tbody>
</table>

Here “$o(b_n)$” would be pronounced “little oh of $b$ sub $n$”, and $\Theta(b_n)$ would be pronounced “Theta of $b$ sub $n$”.

We will not be using the Landau notation, but you ought to be aware that it exists.
EXERCISES

1–8 Evaluate the limit.

1. \( \lim_{n \to \infty} \frac{n^5}{e^n} \)
2. \( \lim_{n \to \infty} \frac{n \sqrt{n}}{\ln n} \)
3. \( \lim_{n \to \infty} \frac{n}{n!} \)
4. \( \lim_{n \to \infty} \frac{2^n}{e^n} \)
5. \( \lim_{n \to \infty} \frac{e^{2n}}{n!} \)
6. \( \lim_{n \to \infty} \frac{\sqrt{n^6}}{n^4} \)
7. \( \lim_{n \to \infty} \frac{\ln(3^n)}{n} \)
8. \( \lim_{n \to \infty} \frac{2^n}{5\sqrt{4^n}} \)

9–14 Evaluate the limit.

9. \( \lim_{n \to \infty} \frac{n^2 - 1}{n^3 + 1} \)
10. \( \lim_{n \to \infty} \frac{\sqrt{n + 2n + 3n^2}}{n^2 + 1} \)
11. \( \lim_{n \to \infty} \frac{(n!)^2 - 1}{n^5 + \sqrt{n}} \)
12. \( \lim_{n \to \infty} \frac{8n^5 + 4n^3 + 3^n}{n^7 + 2^n + 5 \ln n} \)
13. \( \lim_{n \to \infty} \frac{n^2 + 4n + 1}{n + 2^n} \)
14. \( \lim_{n \to \infty} \frac{2^n + 3^n + n^n}{n!} \)

15–22 Evaluate the limit.

15. \( \lim_{n \to \infty} \frac{\sqrt{n^6 + 2n + 1}}{1 - 2n^3} \)
16. \( \lim_{n \to \infty} \frac{5n^{10} + 2}{(n^2 + 1)^5} \)
17. \( \lim_{n \to \infty} \frac{\ln(n^5 + 1)}{1 + \ln n} \)
18. \( \lim_{n \to \infty} \frac{n^2 + \sqrt{4n^3 + 1}}{\ln(n - 3) + 2n^2} \)
19. \( \lim_{n \to \infty} \frac{\sqrt{n + \ln(2^n)}}{5n + \ln(n^2)} \)
20. \( \lim_{n \to \infty} \frac{\sqrt{n^2 + 2n + 5}}{n \ln(n^2 + 1)} \)
21. \( \lim_{n \to \infty} \frac{n^2 + 5\sqrt{9^n + 7}}{e^n + 3^n + 1} \)
22. \( \lim_{n \to \infty} \frac{\sqrt{n + 3n \ln n}}{1 + \ln(n^n + n!)} \)

23–24 Evaluate the limit.

23. \( \lim_{n \to \infty} \frac{e^{n^2 - 1} + n^3}{1 + \sqrt{n} + e^{n^2}} \)
24. \( \lim_{n \to \infty} \frac{\sqrt{9n^2 + n} - \sqrt{n^2}}{4n + 1} \)