Practice Problems: Midterm Exam

1. Evaluate the following limits. You must justify your answers to receive full credit.

(a) \( \lim_{n \to \infty} \frac{3^n}{2n^2 + n!} \)

(b) \( \lim_{n \to \infty} \frac{1 + \sqrt{4n^2 + 3n}}{n + 5} \)

(c) \( \lim_{n \to \infty} \frac{(n^3 + 6)^2 - n^6}{3n^3 + \ln n} \)

(d) \( \lim_{x \to 0} \frac{\ln(1 + x^3) - x^3}{x^6} \)

2. A sequence \( \{a_n\} \) is defined recursively by the formulas

\[ a_1 = 0 \quad \text{and} \quad a_n = \frac{a_{n-1} + 1}{2} \quad \text{for } n \geq 2. \]

(a) Compute \( a_2, a_3, a_4, \) and \( a_5. \)

(b) Find an explicit (non-recursive) formula for \( a_n. \)

(c) Find the sum of the series \( \sum_{n=1}^{\infty} (1 - a_n). \)

3. Find the sum of each of the following series. For parts (c) and (d), your answers must be exact (no decimal approximations).

(a) \( 1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \frac{1}{256} - \cdots \)

(b) \( \sum_{n=1}^{\infty} \frac{1}{n^2 + n} \)

(c) \( \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \frac{1}{11} - \cdots \)

(d) \( \sum_{n=1}^{\infty} \frac{2^n}{n!} \)

4. Evaluate the following integral:

\[ \int_{3}^{\infty} \left( \frac{1}{x^2} + \frac{2}{x^3} + \frac{3}{x^4} + \frac{4}{x^5} + \cdots \right) dx. \]
5. Determine whether each of the following series converges or diverges. You must justify your answer to receive full credit.

(a) \( \sum_{n=1}^{\infty} \frac{1}{n^2 + 3} \)  
(b) \( \sum_{n=1}^{\infty} \frac{1}{2n+1} \)  
(c) \( \sum_{n=1}^{\infty} \frac{n^3}{2^n} \)

(d) \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n+3} \)  
(e) \( \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3} \)  
(f) \( \sum_{n=1}^{\infty} \frac{n!}{2^n} \)

(g) \( \sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n^4 + 1}} \)  
(h) \( \sum_{n=1}^{\infty} \frac{(-1)^n}{1 + e^{-n}} \)  
(i) \( \sum_{n=1}^{\infty} \frac{1}{n \ln(2^n)} \)

6. In the following figure, the large square has a side length of 1, each of the three squares attached to it has a side length of 1/3, and so forth.

Find the total area of all of the squares.

7. (a) Find the sum of the series \( \sum_{n=0}^{\infty} x^{3+4n} \), assuming \(-1 < x < 1\).

(b) Find the radius of convergence of the power series \( \sum_{n=1}^{\infty} \frac{3^n}{n^2} x^n \).

(c) Find a power series representation for the function \( f(x) = x^2 \tan^{-1}(2x) \). Express your answer in summation notation.

(d) Find the first three terms of the Taylor series for the function \( f(x) = \sqrt{x + 25} \) centered at \( a = 0 \).
8. (a) Find the angle between the vectors $\langle -4, 3, 5 \rangle$ and $\langle 3, 4, -5 \rangle$.

(b) Find the area of the parallelogram in $\mathbb{R}^2$ with vertices $(1, 1)$, $(4, 2)$, $(5, 4)$, and $(2, 3)$.

(c) Exactly one pair of the vectors $\langle 2, 0, 1 \rangle$, $\langle 1, -1, 3 \rangle$, $\langle 3, 1, -1 \rangle$, and $\langle 2, 5, 1 \rangle$ are orthogonal. Which pair is it?

(d) Find a vector that is parallel to the plane $3x + 2y + 4z = 7$ and orthogonal to $\langle 2, -1, -1 \rangle$.

9. The following figure shows a pyramid in $\mathbb{R}^3$.

(a) Find an outward-pointing normal vector for the triangular face with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

(b) Find the angle between this face and the square base. Your answer must be correct to the nearest degree.

10. (a) Find the distance between the planes $x + 8y + 4z = 12$ and $x + 8y + 4z = 30$.

(b) Find the equation of the plane containing the lines

$$\begin{align*}
(x, y, z) &= (2, 1 + t, 3 + 2t) \quad \text{and} \quad (x, y, z) = (2 - t, 1 + 4t, 3 + t).
\end{align*}$$

(c) Find the point on the plane $2x + 3y + 6z = 6$ that is closest to the point $(2, 1, 1)$. 
11. The following figure shows a trapezoid in $\mathbb{R}^2$ with two right angles.

Find the coordinates of the point $P$.

12. Let $L$ be the line $(x, y, z) = (2 + 3t, 3 - 2t, 1 + t)$, and let $P$ be the plane $x + 2y + z = 4$. Find the equation of the plane perpendicular to $P$ that contains the line $L$. 