7. Series With Negative Terms

Series with a mix of positive and negative terms can behave very differently than series whose terms are all positive. For a positive series $\sum a_n$, there are only two possibilities:

1. $\sum a_n$ converges, or
2. $\sum a_n = \infty$ (and hence $\sum a_n$ diverges).

As the following example shows, a series with negative terms can diverge without adding up to infinity.

**EXAMPLE 1** Consider the following series:

$$\sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + \cdots$$

The partial sums are:

\[
\begin{align*}
s_1 &= 1 \\
s_2 &= 1 - 1 = 0 \\
s_3 &= 1 - 1 + 1 = 1 \\
s_4 &= 1 - 1 + 1 - 1 = 0 \\
&\vdots
\end{align*}
\]

As you can see, the sequence of partial sums oscillates between 0 and 1, having no limit as $n \to \infty$. Therefore, the series diverges.

Perhaps you are thinking that the sum should be 0, since the 1’s and −1’s cancel in pairs:

\[
\begin{align*}
1 - 1 + 1 - 1 + 1 - 1 + \cdots &= (1 - 1) + (1 - 1) + (1 - 1) + \cdots \\
&= 0 + 0 + 0 + \cdots \\
&= 0
\end{align*}
\]

However, there is just as good an argument that the sum should be 1:

\[
\begin{align*}
1 - 1 + 1 - 1 + 1 - 1 + \cdots &= 1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + \cdots \\
&= 1 + 0 + 0 + 0 + \cdots \\
&= 1
\end{align*}
\]

The point is, there really is no way to make sense of this infinite sum. It is neither infinity nor a number. It just diverges.
Here is a picture of the series in the last example:

As you add the terms of the series together, the partial sums repeatedly jump back and forth (oscillate) between 0 and 1. This is a new kind of divergence: *divergence by oscillation*.

Not every series that “jumps back and forth” diverges. Consider the series:

\[ 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \ldots \]

This is a geometric series with \( r = -\frac{1}{2} \) and \( a = 1 \), so the sum is:

\[ \frac{a}{1 - r} = \frac{1}{1 - (-\frac{1}{2})} = \frac{1}{3/2} = \frac{2}{3} \]

Here is a picture of this series:

As you can see, the partial sums repeatedly jump back and forth across 2/3. The jumps get smaller and smaller, so the series converges to 2/3.
Absolute Convergence

Consider the following series:

\[ 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \cdots \]

This is similar to the series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \), except that every other term has been negated. Do you think this series converges?

Of course it does. If the terms of a series go to zero quickly enough, it shouldn't matter whether they are being added or subtracted. The terms of the above series become small very quickly, which precludes any sort of divergence (including divergence by oscillation).

In general, a series with a mix of positive and negative terms can be transformed into a positive series by changing all the minus signs to plus signs:

\[ 1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \cdots \]

\[ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \cdots \]

\[ 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots \]

This is the same as taking the absolute value of all the terms. If the terms are small enough that the positive series converges, then the original series must converge as well.

**ABSOLUTE CONVERGENCE TEST**

A series \( \sum a_n \) converges absolutely if the associated positive series \( \sum |a_n| \) converges.

Any series that converges absolutely must converge.

**EXAMPLE 2** Determine whether the series \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \) converges or diverges.

**SOLUTION** The \((-1)^n\) causes the signs of the terms to alternate:

\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} = -\frac{1}{1^3} + \frac{1}{2^3} - \frac{1}{3^3} + \frac{1}{4^3} - \cdots \]

This is because \((-1)^n\) is an alternating sequence of 1’s and -1’s:

\[ (-1)^1 = -1 \quad (-1)^2 = 1 \quad (-1)^3 = -1 \quad (-1)^4 = 1 \quad \cdots \]

Therefore, taking the absolute value just eliminates the \((-1)^n\):

\[ \sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots \]

This is a p-series with \( p = 3 \), which is greater than 1. Therefore, it converges.

Therefore, the series \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \) converges.
Since \( \sum_{n=1}^{\infty} \frac{1}{n^3} \) converges, the series \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \) converges absolutely.

It is possible for a series with negative terms to converge even if it does not converge absolutely.

**EXAMPLE 3** Consider the series

\[
1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \cdots
\]

This series does not converge absolutely, since taking the absolute value of each term gives twice the harmonic series. Nonetheless, this series *does* converge. In particular, the first few partial sums are

\[
\begin{align*}
s_1 &= 1 \\
s_2 &= 0 \\
s_3 &= 1/2 \\
s_4 &= 0 \\
s_5 &= 1/3 \\
s_6 &= 0 \\
s_7 &= 1/4 \\
\end{align*}
\]

As you can see, the partial sums converge to zero, and therefore the sum of this series is zero.

Always keep in mind the logical relationship between absolute convergence: every absolutely convergent series converges, but some series that converge do not converge absolutely.

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**The Root Test for General Series**

You can sometimes use the root test in combination with the Absolute Convergence Test to show that a series converges.

**EXAMPLE 4** Determine whether the series \( \sum_{n=1}^{\infty} \frac{(-2)^n}{n^3 e^n} \) converges or diverges.

**SOLUTION** The absolute value of \((-2)^n\) is just \(2^n\):

\[
\sum_{n=1}^{\infty} \left| \frac{(-2)^n}{n^3 e^n} \right| = \sum_{n=1}^{\infty} \frac{2^n}{n^3 e^n}
\]
This series converges by the root test (with \( r = 2/e < 1 \)), and therefore the series \( \sum_{n=1}^{\infty} \frac{(-2)^n}{n^3e^n} \) converges absolutely.

Suppose instead that the last example had been slightly different, so that \( r \) had come out greater than one. Would this prove that the series diverges?

Yes, though this does not follow from the Absolute Convergence Test. It is instead related to an important property of the root test: **if a series has \( r > 1 \), then the individual terms of the series go to infinity.** The reason that a series with \( r > 1 \) diverges is that the terms don’t go to zero, and this is not affected by whether the terms are positive or negative.

THE ROOT TEST (FOR GENERAL SERIES)

Let \( \sum a_n \) be any series, and let:

\[
r = \lim_{n \to \infty} \sqrt[n]{|a_n|}
\]

(a) If \( r < 1 \), then the series \( \sum a_n \) converges absolutely.

(b) If \( r > 1 \), then the series \( \sum a_n \) diverges.

(c) If \( r = 1 \), then the root test is inconclusive.

**Alternating Series**

An **alternating series** is a series whose terms alternate between positive and negative:

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots
\]

Usually an alternating series will have something like a \((-1)^n\) in the formula, since \((-1)^n\) alternates between \(-1\) and \(+1\). (The above series uses \((-1)^{n-1}\), which has the advantage of making the \( n = 1 \) term positive.)

The series above is known as the **alternating harmonic series**. Clearly this series does not converge absolutely. The surprising thing is that it does converge:
Because the series is alternating, the partial sums jump back and forth (left, then right, then left, and so on). Further, each term of the series is smaller than the last, so each jump is shorter than the previous jump. Since the size of the jumps goes to zero, the series eventually narrows in on a limit (indicated by the vertical line).

THE ALTERNATING SERIES TEST

Let $\sum a_n$ be a series. Suppose that:

1. The terms $a_n$ alternate between positive and negative.
2. Each term is smaller than the last, i.e. $|a_1| \geq |a_2| \geq |a_3| \geq \cdots$.
3. The terms go to zero, i.e. $\lim_{n \to \infty} a_n = 0$.

Then the series $\sum a_n$ converges.

EXAMPLE 5  Determine whether the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ converges or diverges.

SOLUTION  Note that $\sum_{n=2}^{\infty} \frac{1}{\ln n} = \infty$, so the given series does not converge absolutely.

But does it converge? The $(-1)^n$ makes it an alternating series:

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n} = \frac{1}{\ln 2} - \frac{1}{\ln 3} + \frac{1}{\ln 4} - \frac{1}{\ln 5} + \cdots$$
Each term is smaller than the last, and the terms go to zero (since $1/\ln n \to 0$ as $n \to \infty$). Therefore, the series converges by the Alternating Series Test.

Because the alternating series test is so powerful, it can be easy to jump to the conclusion that a given alternating series automatically converges. Before you do this, make sure to check the other hypotheses, especially the requirement that the terms themselves converge to zero. For example, the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n+1}$$

may be alternating, but it sure doesn’t converge!

**EXERCISES**

1–18 ■ Determine whether the given alternating series converges or diverges. If it does converge, determine whether it converges absolutely.

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$
2. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/4}}$
3. $\sum_{n=1}^{\infty} \frac{(-1)^n}{3^{1/n}}$
4. $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n^2} + 1}$
5. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{1 + \ln n}$
6. $\sum_{n=0}^{\infty} \frac{(-1)^n}{1 + e^{-n}}$
7. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$
8. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n \sqrt{n}}$
9. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$
10. $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\ln(n^2)}$
11. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n + 3}$
12. $\sum_{n=0}^{\infty} \frac{(-1)^n n}{n^2 + 1}$
13. $\sum_{n=0}^{\infty} \frac{(-1)^n}{1 + \cos^2 n}$
14. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$
15. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\tan^{-1} n}$
16. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{\sqrt{n^6 + 4}}$
17. $\sum_{n=2}^{\infty} \frac{(-1)^n \sqrt{n}}{\ln n}$
18. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^3}$

19–24 ■ Use the root test to determine whether the given series converges or diverges.

19. $\sum_{n=1}^{\infty} \frac{(-2)^n}{n!}$
20. $\sum_{n=1}^{\infty} \frac{(-1)^n n^n}{n 3^n}$
21. $\sum_{n=1}^{\infty} \frac{(-1)^n e^n}{n \sqrt{n + 1}}$
22. $\sum_{n=1}^{\infty} \frac{(3e)^n}{(-2)^n 5^n \sqrt{n}}$
23. $\sum_{n=1}^{\infty} \frac{(-3)^n n^3}{22n + 1}$
24. $\sum_{n=2}^{\infty} \frac{(-1)^n n!}{3^{3n} \ln n}$