1 Some Linear Algebra

These notes briefly discuss some linear algebra relevant to the second derivative test. We begin by reviewing eigenvalues and eigenvectors and then move on to definite and indefinite matrices.

**Definition: Eigenvalues**

Let $A$ be an $n \times n$ matrix. A real or complex number $\lambda$ is called an eigenvalue of $A$ if

$$A \mathbf{v} = \lambda \mathbf{v}$$

for some nonzero vector $\mathbf{v}$.

The vector $\mathbf{v}$ is called an eigenvector for $A$ associated to the eigenvalue $\lambda$. For our purposes, the eigenvectors of a matrix will be much less important than the eigenvalues.

As discussed in a linear algebra course, the eigenvalues of a matrix $A$ are the roots of the characteristic polynomial

$$p(\lambda) = \det(A - \lambda I).$$

The degree of this polynomial is equal to the size of the matrix, and therefore an $n \times n$ matrix has at most $n$ distinct eigenvalues. Indeed, if we include multiple roots of the characteristic polynomial, then an $n \times n$ matrix has exactly $n$ eigenvalues.

The primary motivation for considering eigenvalues is the following.

**Eigenvalues of a Diagonal Matrix**

Let $A$ be a diagonal matrix:

$$A = \begin{bmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \ddots & \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \lambda_n
\end{bmatrix}$$

Then the eigenvalues of $A$ are precisely the diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_n$.

That is, eigenvalues of a matrix are a generalization of the diagonal entries of a diagonal matrix. Indeed, there are many problems in mathematics involving a matrix $A$ that can be solved easily in the case where $A$ is diagonal; in such cases, the solution for an arbitrary matrix $A$ usually involves the eigenvalues of $A$.

**Determinant and Trace**

There is close connection between the eigenvalues of a matrix and its determinant.

**Eigenvalues and the Determinant**

Let $A$ be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then the determinant of $A$ is precisely the product of its eigenvalues:

$$\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n.$$
Similarly, recall that the trace of a matrix is the sum of the entries along the main diagonal. For example, the trace of the matrix
\[
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{bmatrix}
\]
is 1 + 5 + 9 = 15. If \(A\) is an \(n \times n\) matrix, we usually write \(\text{tr}(A)\) for the trace of \(A\).

**Eigenvalues and the Trace**

Let \(A\) be an \(n \times n\) matrix with eigenvalues \(\lambda_1, \lambda_2, \ldots, \lambda_n\). Then the trace of \(A\) is precisely the sum of its eigenvalues:
\[
\text{tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n.
\]

In the case of a \(2 \times 2\) matrix \(A\), this gives us two equations for the eigenvalues:
\[
\lambda_1 + \lambda_2 = \text{tr}(A) \quad \text{and} \quad \lambda_1 \lambda_2 = \det(A),
\]
and we can solve these equations to find \(\lambda_1\) and \(\lambda_2\). This gives us a particularly quick way to find the eigenvalues of a \(2 \times 2\) matrix.

**EXAMPLE 1**

Find the eigenvalues of the matrix
\[
\begin{bmatrix}
1 & -1 \\
2 & 4
\end{bmatrix}
\].

**SOLUTION**
The trace of this matrix is 1 + 4 = 5, and the determinant is\((1)(4) - (-1)(2) = 6\), so
\[
\lambda_1 + \lambda_2 = 5 \quad \text{and} \quad \lambda_1 \lambda_2 = 6.
\]
Then the eigenvalues must be 2 and 3.

**EXAMPLE 2**

Find the eigenvalues of the matrix
\[
\begin{bmatrix}
2 & 3 \\
4 & 1
\end{bmatrix}
\].

**SOLUTION**
The trace of this matrix is 2 + 1 = 3, and the determinant is\((2)(1) - (3)(4) = -10\). Thus
\[
\lambda_1 + \lambda_2 = 3 \quad \text{and} \quad \lambda_1 \lambda_2 = -10,
\]
so the eigenvalues are 5 and -2.

**Symmetric Matrices**

Recall that a square matrix \(A\) is called symmetric if its entries are symmetric across the main diagonal. For example, the matrix
\[
\begin{bmatrix}
5 & 1 & 3 \\
1 & 7 & 4 \\
3 & 4 & 8
\end{bmatrix}
\]

Equivalently, a matrix \(A\) is symmetric if it is equal to its transpose, i.e.
\[
A^T = A.
\]
is symmetric because each of the highlighted entries is equal to the corresponding entry on the other side of the main diagonal.

The eigenvalues of a symmetric matrix have a special property. Unlike most matrices, whose eigenvalues can be either real or complex, the eigenvalues of a symmetric matrix are always real.

### Eigenvalues of a Symmetric Matrix

The eigenvalues of a symmetric matrix are always real numbers.

As a consequence, we can classify symmetric matrices according to the signs of their eigenvalues.

### Definition: Definite and Indefinite Matrices

Let \( A \) be a symmetric matrix.

1. We say that \( A \) is **positive definite** if all of the eigenvalues of \( A \) are positive.
2. We say that \( A \) is **negative definite** if all of the eigenvalues of \( A \) are negative.
3. We say that \( A \) is **indefinite** if \( A \) has at least one positive eigenvalue and at least one negative eigenvalue.

This classification excludes some matrices that have zero as an eigenvalue. For example, a matrix whose eigenvalues are 0, 3, and 5 is not positive definite since 0 is not positive, but is also not indefinite since none of the eigenvalues are negative.

### Example 3

Determine whether the matrix
\[
\begin{bmatrix}
4 & -3 \\
-3 & 12
\end{bmatrix}
\]
is positive definite, negative definite, indefinite, or none of these.

**SOLUTION** The trace of this matrix is 4 + 12 = 16 and the determinant is 39, so
\[
\lambda_1 + \lambda_2 = 16 \quad \text{and} \quad \lambda_1 \lambda_2 = 39.
\]
Then the two eigenvalues must be 3 and 13, so the matrix is positive definite.

### Example 4

Determine whether the matrix
\[
\begin{bmatrix}
1 & 4 \\
4 & 7
\end{bmatrix}
\]
is positive definite, negative definite, indefinite, or none of these.

**SOLUTION** The trace of this matrix is 8, and the determinant is \(-9\). Thus
\[
\lambda_1 + \lambda_2 = 8 \quad \text{and} \quad \lambda_1 \lambda_2 = -9.
\]
Then the two eigenvalues must be \(-1\) and 9, so the matrix is indefinite.
EXERCISES

1–6  ■ Find the eigenvalues of the given matrix.

1. \[
\begin{bmatrix}
8 & 0 \\
0 & 3
\end{bmatrix}
\]

2. \[
\begin{bmatrix}
-2 & 0 & 0 \\
0 & 5 & 0 \\
0 & 0 & 4
\end{bmatrix}
\]

3. \[
\begin{bmatrix}
5 & 2 \\
3 & 6
\end{bmatrix}
\]

4. \[
\begin{bmatrix}
7 & 2 \\
2 & 7
\end{bmatrix}
\]

5. \[
\begin{bmatrix}
-6 & 3 \\
-5 & 2
\end{bmatrix}
\]

6. \[
\begin{bmatrix}
1 & 6 \\
7 & 2
\end{bmatrix}
\]

7–14  ■ Determine whether the given symmetric matrix is positive definite, negative definite, indefinite, or none of these.

7. \[
\begin{bmatrix}
5 & -2 \\
-2 & 5
\end{bmatrix}
\]

8. \[
\begin{bmatrix}
1 & 4 \\
4 & 7
\end{bmatrix}
\]

9. \[
\begin{bmatrix}
-4 & 3 \\
3 & 4
\end{bmatrix}
\]

10. \[
\begin{bmatrix}
-3 & 2 \\
2 & -6
\end{bmatrix}
\]

11. \[
\begin{bmatrix}
5 & 2 \\
2 & 2
\end{bmatrix}
\]

12. \[
\begin{bmatrix}
4 & 6 \\
6 & 9
\end{bmatrix}
\]

13. \[
\begin{bmatrix}
3 & 0 & 0 \\
0 & -5 & 0 \\
0 & 0 & -2
\end{bmatrix}
\]

14. \[
\begin{bmatrix}
0 & 6 \\
6 & 5
\end{bmatrix}
\]