Homework 1 Solutions
Math 318, Spring 2016

Problem 1.

Proposition. Define a sequence \( a_1, a_2, a_3, \ldots \) recursively by \( a_1 = 2 \) and \( a_{n+1} = a_n(a_n + 1) \) for \( n \geq 1 \). Then each \( a_n \) is divisible by at least \( n \) different primes.

Proof. We proceed by induction on \( n \). For the base case \( n = 1 \) observe that \( a_1 \) is divisible by one prime, namely 2. Now suppose that \( a_n \) is divisible by \( n \) different primes \( p_1, \ldots, p_n \). Since \( a_n \) and \( a_n + 1 \) are relatively prime, we know that none of the primes \( p_1, \ldots, p_n \) divide \( a_n + 1 \), so \( a_n + 1 \) must be divisible by at least one prime \( p_{n+1} \) that is distinct from \( p_1, \ldots, p_n \). Then \( a_{n+1} = a_n(a_n + 1) \) is divisible by all of the primes \( p_1, \ldots, p_{n+1} \), so it is divisible by at least \( n + 1 \) different primes. \( \square \)

Problem 2.

Proposition. The set \( P_{3,4} = \{ p \in \mathbb{N} \mid p \text{ is prime and } p \equiv 3 \pmod{4} \} \) is infinite.

Proof. Suppose to the contrary that the set \( P_{3,4} \) is finite, say \( P_{3,4} = \{ p_1, \ldots, p_k \} \), and let

\[ n = 4p_1 \cdots p_k - 1. \]

Note that \( n \) is congruent to 3 modulo 4. But since \( n \) is not divisible by any \( p_i \) and \( n \) is odd, every prime in the prime factorization of \( n \) must be congruent to 1 modulo 4. Since \( n \) is a product of these, \( n \) must itself be congruent to 1 modulo 4, a contradiction. \( \square \)