Problem 2.

Given a pair $a_1, a_2$ of positive integers with $a_1 > a_2$, the corresponding Euclidean sequence $a_1, \ldots, a_n$ is the decreasing sequence defined recursively by the equation

$$a_k = R(a_{k-2}, a_{k-1})$$

where $R(a, b)$ denotes the remainder upon dividing $a$ by $b$. The sequence ends when it reaches zero. For example, the Euclidean sequence starting with $a_1 = 194$ and $a_2 = 57$ is

$$194, 57, 23, 11, 1, 0.$$ 

Proposition. The length of any Euclidean sequence is less than or equal to $1 + 2\lceil \log_2(a_1) \rceil$.

Proof. Let $a_1, \ldots, a_n$ be a Euclidean sequence of length $n$, let $m = \lceil \log_2(a_1) \rceil$, and suppose that $n \geq 2m + 1$. Since $a_k = R(a_{k-2}, a_{k-1})$ for each $k$, we know that

$$a_{k-2} = q a_{k-1} + a_k$$

where $q$ is the integer quotient obtained from dividing $a_{k-2}$ by $a_{k-1}$. Since $a_{k-1} < a_{k-2}$, we know that $q \geq 1$, so

$$a_{k-2} \geq a_{k-1} + a_k > 2a_k$$

for each $k > 2$, and therefore $a_k < a_{k-2}/2$ for each $k > 2$. It follows that

$$a_{2m+1} < \frac{a_1}{2^m} \leq 1,$$

where $a_1 \leq 2^m$ since $\log_2(a_1) \leq m$. But $a_{2m+1}$ is a non-negative integer, so it follows that $a_{2m+1} = 0$, and hence the sequence has length at most $2m + 1$. \qed
Problem 3.

If $a$ is a positive integer and $p$ is a prime, the **multiplicity of $p$ in $a$** is defined by the formula

$$n_p(a) = \max\{k \mid p^k \text{ divides } a\}.$$  

**Part (a)**

**Proposition.** If $a$, $b$, and $c$ are positive integers, then

$$\text{lcm}(a, b, c) = \frac{abc \gcd(a, b, c)}{\gcd(a, b) \gcd(a, c) \gcd(b, c)}.$$  

**Proof.** It suffices to prove that

$$n_p(abc \gcd(a, b, c)) = n_p(\text{lcm}(a, b, c) \gcd(a, b) \gcd(a, c) \gcd(b, c)).$$

for any prime $p$. If we let $A = n_p(a)$, $B = n_p(a)$, and $C = n_p(c)$, then this equation can be written

$$A + B + C + \min(A, B, C) = \max(A, B, C) + \min(A, B) + \min(A, C) + \min(B, C). \quad (*)$$

To prove this, suppose without loss of generality that $A \leq B \leq C$. Then the left side of equation $(*)$ is equal to $A + B + C + A$, and the right side is $C + A + A + B$, so the two sides are equal.

**Part (b)**

**Proposition.** If $a$, $b$, and $c$ are positive integers, then

$$\text{lcm}(a, \gcd(b, c)) = \gcd(\text{lcm}(a, b), \text{lcm}(a, c))$$  

**Proof.** It suffices to prove that

$$n_p(\text{lcm}(a, \gcd(b, c))) = n_p(\gcd(\text{lcm}(a, b), \text{lcm}(a, c)))$$

for any prime $p$. If we let $A = n_p(a)$, $B = n_p(a)$, and $C = n_p(c)$, then this equation can be written

$$\max(A, \min(B, C)) = \min(\max(A, B), \max(A, C)). \quad (*)$$

To prove this, suppose without loss of generality that $B \leq C$. Then the left side of equation $(*)$ is simply $\max(A, B)$. But since $\max(A, B) \leq \max(A, C)$, the right side is also equal to $\max(A, B)$, so the two sides are equal.