Practice Problems: Cyclic Groups
Math 332, Spring 2013

These are not to be handed in. The quiz will be on Tuesday.

1. Find all generators of \( \mathbb{Z}_6 \), \( \mathbb{Z}_8 \), and \( \mathbb{Z}_{20} \).

2. List all elements of the subgroup \( \langle 30 \rangle \) in \( \mathbb{Z}_{80} \).

3. If \( |a| = 60 \), what is the order of \( a^{24} \)?

4. How many subgroups does \( \mathbb{Z}_{20} \) have? List the possible generators for each subgroup.

5. Suppose that \( |a| = 24 \). Find a generator for \( \langle a^{21} \rangle \cap \langle a^{10} \rangle \).

6. Let \( G \) be a cyclic group of order 720. How many subgroups does \( G \) have?

7. How many generators does \( \mathbb{Z}_{125} \) have?

8. In \( \mathbb{Z}_{60} \), list all generators for the subgroup of order 12.

9. Let \( G \) be a group and let \( a \) be an element of \( G \).
   a. If \( a^{10} = e \), what can we say about the order of \( a \)?
   b. If \( a^m = e \), what can we say about the order of \( a \)?

10. List all elements of order 8 in \( \mathbb{Z}_{8000000} \).

11. Determine the subgroup lattice for \( \mathbb{Z}_{p^2q} \), where \( p \) and \( q \) are distinct primes.

12. Determine the subgroup lattice for \( \mathbb{Z}_{p^n} \), where \( p \) is a prime and \( n \) is some positive integer.

13. If \( |x| = 40 \), list all elements of \( \langle x \rangle \) that have order 10.

14. Determine the orders of the elements of \( D_{33} \) and how many there are of each.

15. If \( |a^5| = 12 \), what are the possibilities for \( |a| \)? What if \( |a^5| = 15 \)?
Answers

1. For $\mathbb{Z}_6$, generators are 1 and 5; for $\mathbb{Z}_8$, generators are 1, 3, 5, and 7; for $\mathbb{Z}_{20}$, generators are 1, 3, 7, 9, 11, 13, 17, and 19.

2. 0, 10, 20, 30, 40, 50, 60, 70

3. Since $\langle a^{24} \rangle = \langle a^{12} \rangle$ is a subgroup of order 5, the element $a^{24}$ must have order 5 as well.

4. Six subgroups: $\mathbb{Z}_{20}$ (generated by 1, 3, 7, 9, 11, 13, or 19), the subgroup of even numbers (generated by 2, 6, 14, or 18), the subgroup of multiples of 4 (generated by 4, 8, 12, or 16), the subgroup of multiples of 5 (generated by 5 or 15), the subgroup of multiples of 10 (generated by 10), and the trivial subgroup (generated by 0).

5. $\langle a^{21} \rangle \cap \langle a^{10} \rangle = \langle a^6 \rangle$.

6. 30

7. 100

8. 5, 25, 35, and 55.

9. a. $|a|$ is 1, 2, 5, or 10. b. $|a|$ divides $m$.

10. 1000000, 3000000, 5000000, 7000000.

11.

12.

13. $x^4$, $x^{12}$, $x^{28}$, $x^{36}$.

14. 33 of order 2, 20 of order 33, 10 of order 11, 2 of order 3, one of order 1.

15. $|a| = 12$ or $|a| = 60$; $|a| = 75$. 