Problem 1.
Let \( G = (0, \infty) \times \mathbb{R} \), and let \(*\) be the binary operation on \( G \) defined by
\[
(a, b) \ast (c, d) = (ac, bc + d)
\]
for all \((a, b), (c, d) \in G\).

**Proposition.** \( G \) forms a group under the operation \(*\).

**Proof.** Note first that \( G \) is closed under \(*\). We must show that \(*\) is associative and that \( G \) has an identity element and inverses.

To prove that \(*\) is associative, let \((a, b), (c, d), (e, f) \in G\). Then
\[
\left( ((a, b) \ast (c, d)) \ast (e, f) \right) = (ac, bc + d) \ast (e, f) = (ace, (bc + d)e + f)
\]
and
\[
(a, b) \ast ((c, d) \ast (e, f)) = (a, b) \ast (ce, de + f) = (ace, bce + de + f)
\]
Since the two results are the same, the operation \(*\) is associative.

Next, observe that \((1, 0)\) is an identity element for \( G \), since
\[
(1, 0) \ast (a, b) = (1a, 0a + b) = (a, b)
\]
and
\[
(a, b) \ast (1, 0) = (a(1), b(1) + 0) = (a, b)
\]
for all \((a, b) \in G\).

For inverses, let \((a, b) \in G\), and consider the element \((\frac{1}{a}, -\frac{b}{a}) \in G\). We have
\[
\left( \frac{1}{a}, -\frac{b}{a} \right) \ast (a, b) = \left( \left( \frac{1}{a} \right)a, \left( -\frac{b}{a} \right)a + b \right) = (1, 0)
\]
and
\[
(a, b) \ast \left( \frac{1}{a}, -\frac{b}{a} \right) = \left( a\left( \frac{1}{a} \right), b\left( \frac{1}{a} \right) + \left( -\frac{b}{a} \right) \right) = (1, 0),
\]
so \((\frac{1}{a}, -\frac{b}{a})\) is an inverse for \((a, b)\). We conclude that \( G \) is a group. □
Problem 2.

Theorem. Every group with an even number of elements has at least one element of order two.

Proof. Let $G$ be a group of even order. Then $G - \{e\}$ is closed under inverses, and has an odd number of elements. If we remove each inverse pair of elements, there will be at least one element $g \in G - \{e\}$ left over, which must be its own inverse. Then $g^2 = gg^{-1} = e$, so $g$ has order two.

\[\square\]

Problem 3.

Let $G$ be a group. Given elements $a, b \in G$, we say that $b$ is conjugate to $a$ if there exists an element $c \in G$ such that $b = c^{-1}ac$.

(a) Proposition. Conjugacy is an equivalence relation on $G$.

Proof. We check that conjugacy is reflexive, symmetric, and transitive.

Reflexive. If $a \in G$, then $a = e^{-1}ae$, so $a$ is conjugate to $a$.

Symmetric. Let $a, b \in G$ and suppose that $a$ is conjugate to $b$. Then $a = c^{-1}bc$ for some $c \in G$. Then $b = cac^{-1} = (c^{-1})^{-1}a(c^{-1})$, so $b$ is conjugate to $a$.

Transitive Let $x, y, z \in G$, and suppose that $x$ is conjugate to $y$ and $y$ is conjugate to $z$. Then $x = c^{-1}yc$ and $y = d^{-1}zd$ for some $c, d \in G$. Then $x = c^{-1}d^{-1}zd = (dc)^{-1}z(dc)$, so $x$ is conjugate to $z$.

\[\square\]

(b) The conjugacy classes for $D_3$ are $\{e\}$, $\{r, r^2\}$, and $\{s, rs, r^2s\}$.

(c) The conjugacy classes for $D_4$ are $\{e\}$, $\{r, r^3\}$, $\{r^2\}$, $\{s, r^2s\}$, and $\{rs, r^3s\}$.

Problem 4.

Theorem. Let $G$ be a group, and suppose that $a^2 = e$ for every element $a \in G$. Then $G$ is abelian.

Proof. Note that $a^{-1} = a$ for every element $a \in G$. If $a, b \in G$, it follows that

\[ab = (ab)^{-1} = b^{-1}a^{-1} = ba.\]

\[\square\]