Matrix Groups

Among the most important examples of groups are groups of matrices. The textbook briefly discusses groups of matrices in Chapter 2, and then largely forgets about them. These notes remedy this omission.

Matrix Groups over Fields

As we have seen, it is possible to consider matrices whose entries are elements of any field F. Common choices for F include the field $\mathbb{R}$ of real numbers, the field $\mathbb{C}$ of complex numbers, and finite fields such as $\mathbb{Z}_p$ for p prime.

Definition: Matrix Group
A **matrix group** over a field F is a set of invertible matrices with entries in F that forms a group under matrix multiplication.

Note that the matrices in a matrix group must be square (to be invertible), and must all have the same size. Thus there are 2×2 matrix groups, 3×3 matrix groups, 4×4 matrix groups, and so forth. The size of the matrices is sometimes referred to as the **degree** of the matrix group.

Because the group operation for a matrix group is matrix multiplication, the identity element of a matrix group is always the $n \times n$ identity matrix, and inverses in a matrix group are just the usual inverse matrices.

The most important matrix groups are the general linear groups.

Definition: General Linear Group
Let F be a field, and let $n \in \mathbb{N}$. The **general linear group** of degree n over F, denoted $GL(n, F)$, is the group of all invertible $n \times n$ matrices with entries in F.

Since a matrix is invertible if and only if its determinant is nonzero, $GL(n, F)$ can also be defined as the group of all $n \times n$ matrices with entries in F having nonzero determinant. (Note that the determinant of a matrix with entries in F is by definition an element of F.)

EXAMPLE 1 $GL(2, \mathbb{Z}_2)$

If F is an infinite field such as $\mathbb{R}$ or $\mathbb{C}$, then the general linear group $GL(n, F)$ has infinite order. However, if F is a finite field, then $GL(n, F)$ is a finite group.

For example, consider all 2×2 matrices over the field $\mathbb{Z}_2$. There are sixteen such matrices, since each of the four entries can be either 0 or 1. Of these sixteen matrices, exactly six of them have nonzero determinant:

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix},\quad \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix},\quad \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix},\quad \begin{bmatrix}0 & 1\\1 & 1\end{bmatrix},\quad \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix},\quad \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}.$$

Thus $GL(2, \mathbb{Z}_2)$ is a six-element group. It is easy to check that $GL(2, \mathbb{Z}_2) \approx S_3$. ■

The general linear group $GL(n, F)$ is the most general matrix group, in the same way that S_n is the most general permutation group. In particular, every matrix group is just a subgroup of some $GL(n, F)$.

The most important subgroup of $GL(n, F)$ is the special linear group.

Definition: The Special Linear Group

Let F be a field, and let $n \in \mathbb{N}$. The **special linear group** of degree n over F is defined as follows

$$SL(n, F) \;=\; \{A \in GL(n, F) \mid \det(A) = 1\}.$$

Note that $SL(n, F)$ is indeed a subgroup of $GL(n, F)$. In particular, it is always nonempty (since the identity matrix has determinant 1), and it is closed under multiplication and inverses.

The following examples discuss a few other important matrix groups.

EXAMPLE 2 Diagonal Matrices

A **diagonal matrix** is a square matrix whose nonzero entries are all along the main diagonal:

$$\begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_n\end{bmatrix}$$

It is not hard to see that the set of all diagonal matrices in $GL(n, F)$ forms a subgroup. We shall verify this using the Two-Step Subgroup Test.

1. First, observe that the product of two diagonal matrices is again diagonal, with

$$\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu_n \end{bmatrix} = \begin{bmatrix} \lambda_1\mu_1 & 0 & \cdots & 0 \\ 0 & \lambda_2\mu_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n\mu_n \end{bmatrix}$$

   Thus the set of diagonal matrices is closed under multiplication.

2. Similarly, the inverse of a diagonal matrix is obtained by taking the multiplicative inverse of each diagonal entry:

$$\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}^{-1} = \begin{bmatrix} \lambda_1^{-1} & 0 & \cdots & 0 \\ 0 & \lambda_2^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^{-1} \end{bmatrix}.$$

   Thus the set of diagonal matrices is closed under inverses. ■

EXAMPLE 3 Triangular Matrices

An **upper triangular matrix** is an $n \times n$ matrix that has zeroes below the main diagonal:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}$$

Such a matrix is invertible if and only if all of its diagonal entries are nonzero.

Though it is slightly difficult to prove, the inverse of an upper triangular matrix is upper triangular, and the product of two upper triangular matrices is again upper triangular. Thus the set of all upper triangular matrices in $GL(n, F)$ form a matrix group.

The transpose of an upper triangular matrix is called a **lower triangular** matrix. Everything that works for upper triangular matrices works just as well for lower triangular matrices. In particular, the set of all lower triangular matrices in $GL(n, F)$ also form a matrix group. ■

Linear Transformations

For the remainder of these notes, we will only be considering vectors and matrices over the real numbers.

Let $\mathbb{R}^n$ denote n-dimensional Euclidean space, i.e. the set of all column vectors with n components. For example, $\mathbb{R}^2$ is the Euclidean plane, and $\mathbb{R}^3$ is three-dimensional Euclidean space.

If A is an $n \times n$ matrix and $\mathbf{v} \in \mathbb{R}^n$, the product $A\mathbf{v}$ is again an element of $\mathbb{R}^n$. This lets us think of any $n \times n$ matrix as a function from $\mathbb{R}^n$ to itself.

> **Definition: Linear Transformation**
> Let A be an $n \times n$ matrix. The **linear transformation** associated with A is the function $T\colon \mathbb{R}^n \to \mathbb{R}^n$ defined by
> $$T(\mathbf{v}) \;=\; A\mathbf{v}$$
> for all $\mathbf{v} \in \mathbb{R}^n$.

EXAMPLE 4 Linear Transformations of $\mathbb{R}^2$

Consider the following three matrices:

$$A_1 \;=\; \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad A_2 \;=\; \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \qquad A_3 \;=\; \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

These matrices define three different linear transformations of $\mathbb{R}^2$.

The transformation $T_1\colon \mathbb{R}^2 \to \mathbb{R}^2$ corresponding to A_1 is given by the formula

$$T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \;=\; \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \;=\; \begin{bmatrix} x \\ -y \end{bmatrix}.$$

This transformation negates the y-coordinate of each point, and therefore reflects the plane $\mathbb{R}^2$ across the x-axis.

The transformation $T_2\colon \mathbb{R}^2 \to \mathbb{R}^2$ corresponding to A_2 is given by the formula

$$T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \;=\; \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \;=\; \begin{bmatrix} 2x \\ 2y \end{bmatrix}.$$

Geometrically, this transformation scales each vector by a factor of 2. Therefore, T_2 corresponds to a dilation of the plane by a factor of 2 centered at the origin.

The transformation $T_3\colon \mathbb{R}^2 \to \mathbb{R}^2$ corresponding to A_3 is given by the formula

$$T_3\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \;=\; \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \;=\; \begin{bmatrix} -y \\ x \end{bmatrix}.$$

It is easy to check that this is a 90° counterclockwise rotation of the plane around the origin. ■

You may be surprised that the reflection, dilation, and rotation in the last example were linear transformations. This brings up several questions. First, is every reflection, dilation, or rotation of the plane a linear transformation? And what about other sorts of transformations, such as translations?

The following theorem places some restriction on the possible linear transformations.

Theorem 1 Zero is Fixed

If $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, then $T(\mathbf{0}) = \mathbf{0}$, where $\mathbf{0}$ denotes the zero vector in $\mathbb{R}^n$.

PROOF Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, and let A be the corresponding $n \times n$ matrix. Then $T(\mathbf{v}) = A\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^n$, so $T(\mathbf{0}) = A\mathbf{0} = \mathbf{0}$. ■

Geometrically, this theorem says that any linear transformation must fix the origin in $\mathbb{R}^n$. Thus a translation cannot a linear transformation, and a reflection, dilation, or rotation can only be a linear transformation if it fixes the origin.

Our next theorem gives a complete geometric classification of linear transformations. This is a surprisingly difficult theorem to prove, and we shall not attempt to do so here.

Theorem 2 Geometric Characterization

Let $n \geq 2$, and let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be any function. Then T is a linear transformation if and only if it satisfies the following conditions:

1. *T fixes the origin, i.e. $T(\mathbf{0}) = \mathbf{0}$.*

2. *For every straight line $L \subseteq \mathbb{R}^n$, the image of L under T is also a straight line.*

Using this theorem, we can immediately see that many familiar geometric transformations are in fact linear transformations. These include:

- Any reflection of $\mathbb{R}^2$ across a line through the origin.
- Any reflection of $\mathbb{R}^3$ across a plane through the origin, or more generally any reflection of $\mathbb{R}^n$ across an $(n-1)$-dimensional subspace.

- Any rotation of $\mathbb{R}^2$ about the origin.
- Any rotation of $\mathbb{R}^3$ that fixes the origin.
- Any dilation of $\mathbb{R}^n$ centered at the origin.

Transformation Groups

As we have seen, some of the most important geometric transformations of $\mathbb{R}^n$ are linear transformations. As a result, the symmetry group of a geometric object can sometimes be thought of as a group of linear transformations.

Definition: Transformation Group
A **transformation group** is a group whose elements are linear transformations, and whose operation is composition.

We have already seen a few examples of transformations groups:

- The dihedral groups are transformation groups. In particular let P be a regular n-gon in the plane centered at the origin. Then each rotation or reflection of P is a linear transformation, so the resulting dihedral group D_n is a transformation group.
- Similarly, let P be any polyhedron in $\mathbb{R}^3$ (such as a tetrahedron or cube) centered at the origin. Then every symmetry of P is a linear transformation, so the symmetry group of P is a transformation group.

The following theorem can be very helpful for working with transformation groups.

Theorem 3 Compositions and Inverses of Transformations

Let A and B be $n \times n$ matrices, and let T_A and T_B be the corresponding linear transformations. Then:

1. *The composition $T_A \circ T_B$ is a linear transformation, corresponding to the matrix AB.*
2. *If A is invertible, then T_A is bijective, and $(T_A)^{-1}$ is the linear transformation corresponding to the matrix A^{-1}.*

PROOF Recall that $T_A(\mathbf{v}) = A\mathbf{v}$ and $T_B(\mathbf{v}) = B\mathbf{v}$ for any vector $\mathbf{v} \in \mathbb{R}^n$. Then

$$(T_A \circ T_B)(\mathbf{v}) \;=\; T_A(T_B(\mathbf{v})) \;=\; T_A(B\mathbf{v}) \;=\; A(B\mathbf{v}) \;=\; (AB)\mathbf{v}$$

for all $\mathbf{v} \in \mathbb{R}^n$, which proves that $T_A \circ T_B$ is the linear transformation corresponding to AB.

For part (2), let $T_{A^{-1}}$ be the linear transformation corresponding to A^{-1}, and let T_I be the linear transformation corresponding to the $n \times n$ identity matrix. Since $AA^{-1} = A^{-1}A = I$, it follows from part (1) that $T_A \circ T_{A^{-1}} = T_{A^{-1}} \circ T_A = T_I$. But T_I is the identity function on $\mathbb{R}^n$, and therefore $T_{A^{-1}}$ an the inverse function for T_A. ■

This theorem lets us work with transformation groups as though they were groups of matrices. Indeed, every transformation group is isomorphic to a group of matrices.

Corollary 4 Matrix Representation of Transformation Groups

Let G be a transformation group, and let H be the corresponding set of $n \times n$ matrices. Then H is a subgroup of $GL(n,\mathbb{R})$, and G and H are isomorphic.

PROOF Left to the reader. ■

EXAMPLE 5 A Dihedral Group

Let D_4 be the group of symmetries of the square $[-1,1] \times [-1,1]$ in $\mathbb{R}^2$. Recall that

$$D_4 \;=\; \{e, r, r^2, r^3, s, rs, r^2s, r^3s\},$$

where r is a 90° counterclockwise rotation about the origin, and s is a reflection across the x-axis. Note that each of the elements of D_4 is a linear transformation, and therefore D_4 is a transformation group.

As seen Example 4, the transformations r and s correspond to the following pair of matrices:

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

By multiplying R and S, we can find matrices corresponding to each of the elements of D_4:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad R^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad R^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad RS = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad R^2S = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad R^3S = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

In particular, each of the matrices R, R^2, R^3 corresponds to a rotation of the plane, while each of the matrices S, RS, R^2S, R^3S corresponds to a reflection of the plane across a certain line through the origin. Together, these eight matrices form a subgroup of $GL(2,\mathbb{R})$ that is isomorphic to D_4. ■

Finding the Matrix for a Transformation

For our matrix representation approach to be useful, we need a method to find the matrix that corresponds to a given linear transformation. We shall use a certain trick which is often useful for finding the columns of an unknown matrix.

Recall that the **standard basis vectors** for $\mathbb{R}^n$, denoted $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are the columns of the $n \times n$ identity matrix:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

The trick we will use is this: if A is any matrix and $\mathbf{e}_i$ is a standard basis vector, then the product $A\mathbf{e}_i$ is just the i'th column of A. For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}.$$

Theorem 5 Formula for the Matrix of a Linear Transformation

Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Then the columns of the matrix corresponding to T are the vectors

$$T(\mathbf{e}_1), \quad T(\mathbf{e}_2), \quad \ldots, \quad T(\mathbf{e}_n),$$

where $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ denote the standard basis vectors for $\mathbb{R}^n$.

PROOF Let A be the $n \times n$ matrix corresponding to T, so that $T(\mathbf{v}) = A\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^n$. Then $T(\mathbf{e}_i) = A\mathbf{e}_i$ for each standard basis vector $\mathbf{e}_i$. But $A\mathbf{e}_i$ is just the i'th column of A, and therefore the i'th column of A is $T(\mathbf{e}_i)$. ■

EXAMPLE 6 The Matrix for a Reflection

Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be reflection across the line $y = -x$. It is clear from the geometry

that
$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}0\\-1\end{bmatrix}\qquad\text{and}\qquad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}-1\\0\end{bmatrix},$$
so the matrix for T is $\begin{bmatrix}0&-1\\-1&0\end{bmatrix}$. ■

EXAMPLE 7 The Matrix for a Rotation

Let $T\colon \mathbb{R}^2\to\mathbb{R}^2$ be a counterclockwise rotation about the origin by an angle of θ. A little basic trigonometry shows that
$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}\cos\theta\\\sin\theta\end{bmatrix}\qquad\text{and}\qquad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}\cos(\theta+90^\circ)\\\sin(\theta+90^\circ)\end{bmatrix}=\begin{bmatrix}-\sin\theta\\\cos\theta\end{bmatrix}$$
Therefore, the matrix for T is $\begin{bmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{bmatrix}$. ■

The method of Theorem 5 also works for linear transformations in three dimensions.

EXAMPLE 8 A Rotation in Three Dimensions

Let L be the line $y=x$ in the xy-plane, and let T be the 180° rotation of $\mathbb{R}^3$ about the line L. It is not hard to see that
$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}0\\1\\0\end{bmatrix},\qquad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}1\\0\\0\end{bmatrix},\qquad\text{and}\qquad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}0\\0\\-1\end{bmatrix},$$
so the matrix for T is $\begin{bmatrix}0&1&0\\1&0&0\\0&0&-1\end{bmatrix}$. ■

Using the method of Theorem 5, we can find matrices corresponding to the elements of any transformation group. In most cases, the fastest approach is to use Theorem 5 to find matrices corresponding to the elements of a generating set, and then multiply to obtain the remaining matrices.

EXAMPLE 9 Dihedral Group of a Triangle

Let D_3 be the symmetry group of an equilateral triangle centered at the origin of $\mathbb{R}^2$, with one of its vertices along the x-axis. Then
$$D_3 \;=\; \{e,r,r^2,s,rs,r^2s\},$$

where r is a 120° counterclockwise rotation about the origin, and s is a reflection across the x-axis. According to the formula in Example 7, the matrix corresponding to r is

$$R \;=\; \begin{bmatrix} \cos 120^\circ & -\sin 120^\circ \\ \sin 120^\circ & \cos 120^\circ \end{bmatrix} \;=\; \frac{1}{2}\begin{bmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}.$$

Combining this with the matrix $S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, we obtain the following six matrices:

$$I \;=\; \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad R \;=\; \frac{1}{2}\begin{bmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} \qquad R^2 \;=\; \frac{1}{2}\begin{bmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad RS \;=\; \frac{1}{2}\begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} \qquad R^2S \;=\; \frac{1}{2}\begin{bmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}$$

This six-element matrix group is isomorphic to the dihedral group D_3. ■

EXAMPLE 10 Rotations of a Cube

Let G be the group of rotations of the cube $[-1,1] \times [-1,1] \times [-1,1]$ in $\mathbb{R}^3$. Note that, among other things, G includes 90° rotations about the x and y axes. The corresponding 3×3 matrices are

$$R_x \;=\; \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \text{and} \qquad R_y \;=\; \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

Using matrix multiplication, we can combine R_x and R_y to obtain the group of 24 possible rotations of the cube:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \qquad ■$$