Sample Homework Solutions

Problem 1.

Proposition. If \( n \) is a positive integer, then
\[
\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.
\]

Proof. We proceed by induction on \( n \). For \( n = 1 \), we have
\[
\sum_{i=1}^{1} i = 1 = \frac{1(1+1)}{2}.
\]

Now suppose that the proposition holds for \( n \). Then
\[
\sum_{i=1}^{n+1} i = \left( \sum_{i=1}^{n} i \right) + (n+1) = \frac{n(n+1)}{2} + (n+1)
\]
\[
= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} = \frac{(n+2)(n+1)}{2},
\]
so the proposition holds for \( n + 1 \) as well. Therefore, by induction, the statement holds for all positive integers \( n \). \qed

Problem 2.

Proposition. If \( S \) is a set, there does not exist any surjection \( S \to \mathcal{P}(S) \).

Proof. Suppose to the contrary that \( f : S \to \mathcal{P}(S) \) were a surjection, and consider the set
\[
\Delta = \{ s \in S : s \not\in f(s) \}.
\]
Then \( \Delta \in \mathcal{P}(S) \), so by hypothesis \( \Delta = f(\delta) \) for some element \( \delta \in S \). But
\[
\delta \in \Delta \iff \delta \not\in f(\delta) \quad \text{(by the definition of } \Delta),
\]
\[
\iff \delta \not\in \Delta \quad \text{(since } f(\delta) = \Delta),
\]
a contradiction. Therefore, such a surjection \( f \) cannot exist. \qed
Problem 3.

(a) For a function \(\{1, 2\} \to \{1, 2, 3, 4, 5\}\) each of the elements 1, 2 has five possible images. Therefore, there are \(5 \times 5 = 25\) such functions.

(b) For an injection \(\{1, 2\} \to \{1, 2, 3, 4, 5\}\), the element 1 has five possible images. Once the image of 1 is determined, the element 2 has only four possible images, since the image of 2 cannot be the same as the image of 1. Therefore, there are a total of \(5 \times 4 = 20\) possible injections.

(c) As in part (a), each of the elements 1, 2, 3 has two possible images, for a total of \(2 \times 2 \times 2 = 8\) possible functions. Of these functions, two are not surjective, namely the constant function 1 and the constant function 2. It follows that there are \(8 - 2 = 6\) possible surjections.