Quiz 1 Practice Problems: Cyclic Groups
Math 332, Spring 2010

These are not to be handed in. The quiz will be on Tuesday.

1. Find all generators of $Z_6$, $Z_8$, and $Z_{20}$.

2. List all elements of the subgroup $\langle 30 \rangle$ in $Z_{80}$.

3. If $|a| = 60$, what is the order of $a^{24}$?

4. How many subgroups does $Z_{20}$ have? List the possible generators for each subgroup.

5. Suppose that $|a| = 24$. Find a generator for $\langle a^{21} \rangle \cap \langle a^{10} \rangle$. In general, what is a generator for the subgroup $\langle a^m \rangle \cap \langle a^n \rangle$?

6. In $Z_{60}$, list all generators for the subgroup of order 12.

7. Let $G$ be a group and let $a$ be an element of $G$.
   a. If $a^{12} = e$, what can we say about the order of $a$?
   b. If $a^m = e$, what can we say about the order of $a$?

8. List all elements of order 8 in $Z_{8000000}$. How do you know your list is complete?

9. Determine the subgroup lattice for $Z_{p^2q}$, where $p$ and $q$ are distinct primes.

10. Determine the subgroup lattice for $Z_{p^n}$, where $p$ is a prime and $n$ is some positive integer.

11. If $|x| = 40$, list all elements of $\langle x \rangle$ that have order 10.

12. Determine the orders of the elements of $D_{33}$ and how many there are of each.

13. If $|a^5| = 12$, what are the possibilities for $|a|$? What if $|a^5| = 15$?
Answers

1. For \( Z_6 \), generators are 1 and 5; for \( Z_8 \), generators are 1, 3, 5, and 7; for \( Z_{20} \), generators are 1, 3, 7, 9, 11, 13, 17, and 19.

2. 0, 10, 20, 30, 40, 50, 60, 70

3. Since \( \langle a^{24} \rangle = \langle a^{12} \rangle \) is a subgroup of order 5, the element \( a^{24} \) must have order 5 as well.

4. Six subgroups: \( Z_{20} \) (generated by 1, 3, 7, 9, 11, 13, 17, or 19), the subgroup of even numbers (generated by 2, 6, 14, or 18), the subgroup of multiples of 4 (generated by 4, 8, 12, or 16), the subgroup of multiples of 5 (generated by 5 or 15), the subgroup of multiples of 10 (generated by 10), and the trivial subgroup (generated by 0).

5. \( \langle a^{21} \rangle \cap \langle a^{10} \rangle = \langle a^6 \rangle \). In the general case \( \langle a^m \rangle \cap \langle a^n \rangle = \langle a^k \rangle \), where \( k = \text{lcm}(m, n) \mod 24 \).

6. 5, 25, 35, and 55.

7. a. \( |a| \) divides 12. b. \( |a| \) divides \( m \).

8. 1000000, 3000000, 5000000, 7000000. By Theorem 4.3, \( \langle 1000000 \rangle \) is the unique subgroup of order 8, and only those on the list are generators

9.

\[
\begin{array}{c}
\langle 1 \rangle \\
\langle p \rangle \\
\langle q \rangle \\
\langle p^2 \rangle \\
\langle pq \rangle \\
\langle 0 \rangle
\end{array}
\quad \text{or} \quad
\begin{array}{c}
Z_{p^2 q} \\
Z_{pq} \\
Z_p
\end{array}
\]

10.

\[
\begin{array}{c}
\langle 1 \rangle \\
\langle p \rangle \\
\langle p^2 \rangle \\
\vdots \\
\langle p^{n-1} \rangle \\
\langle 0 \rangle
\end{array}
\quad \text{or} \quad
\begin{array}{c}
Z_p^n \\
Z_{p^{n-1}} \\
Z_{p^{n-2}} \\
\vdots \\
Z_p
\end{array}
\]

11. \( x^4, x^{12}, x^{28}, x^{36} \).

12. 33 of order 2, 20 of order 33, 10 of order 11, 2 of order 3, one of order 1.

13. \( |a| = 12 \) or \( |a| = 60; |a| = 75 \).