Homework 4 Solutions

Problem 13.4.

Proposition. If \( \{ T_\alpha \} \) is a family of topologies on \( X \), then \( \bigcap T_\alpha \) is a topology on \( X \).

Proof. Since \( \emptyset \) and \( X \) are in each \( T_\alpha \), they must also be elements of \( \bigcap T_\alpha \). Next, if \( C \subset \bigcap T_\alpha \) for all \( \alpha \), then \( C \subset T_\alpha \) for all \( \alpha \); it follows that \( \bigcup C \in T_\alpha \) for all \( \alpha \), and hence \( \bigcup C \in \bigcap T_\alpha \). Finally, if \( C \) is a finite subcollection of \( \bigcap T_\alpha \), then \( \bigcap C \in T_\alpha \) for all \( \alpha \), and therefore \( \bigcap C \in \bigcap T_\alpha \). \( \Box \)

It is not true in general that the union of two topologies is a topology. For example, the union \( T_1 \cup T_2 = \{ \emptyset, X, \{ a \}, \{ a, b \}, \{ b, c \}, \{ a, b \}, \{ b, c \}, \} \) of the two topologies from part (c) is not a topology, since \( \{ a, b \}, \{ b, c \} \in T_1 \cup T_2 \) but \( \{ a, b \} \cap \{ b, c \} = \{ b \} \notin T_1 \cup T_2 \).

Proposition. Let \( \{ T_\alpha \} \) be a family of topologies on \( X \). Then there exists a unique smallest topology on \( X \) containing all the collections \( T_\alpha \), and a unique largest topology contained in all \( T_\alpha \).

Proof. The unique largest topology contained in all the \( T_\alpha \) is simply the intersection \( \bigcap T_\alpha \). For the other statement, observe that the family of all topologies on \( X \) that contain \( \bigcup T_\alpha \) is nonempty, since it includes the discrete topology on \( X \). Then the intersection of this family is the unique smallest topology that contains every \( T_\alpha \). \( \Box \)

For part (c), the smallest topology containing both \( T_1 \) and \( T_2 \) is

\[ \{ \emptyset, X, \{ a \}, \{ b \}, \{ a, b \}, \{ b, c \} \}, \]

and the largest topology contained in \( T_1 \) and \( T_2 \) is the intersection

\[ T_1 \cap T_2 = \{ \emptyset, X, \{ a \} \}. \]
Problem 13.8.

**Proposition.** The countable collection $\mathcal{B} = \{(a,b) \mid a < b, \text{a and b rational}\}$ is a basis for the standard topology on $\mathbb{R}$.

*Proof.* Let $U \subset \mathbb{R}$ be open, and let $x \in U$. Since the open intervals are a basis for the topology on $\mathbb{R}$, there exists an open interval $(a,b)$ such that $x \in (a,b) \subset U$. Then $a < x < b$, so there exist rational numbers $q$ and $r$ such that $a < q < x < r < b$. Then $(q,r) \in \mathcal{B}$ and $x \in (q,r) \subset U$, so $\mathcal{B}$ is a basis for the standard topology on $\mathbb{R}$ by Lemma 13.2. \hfill $\square$

**Proposition.** The basis $\mathcal{C} = \{[a,b) \mid a < b, \text{a and b rational}\}$ generates a topology on $\mathbb{R}$ different from the lower limit topology.

*Proof.* The set $[\sqrt{2},2)$ is open in the lower-limit topology. However, since $\sqrt{2}$ is irrational, there does not exist an $[a,b) \in \mathcal{C}$ for which $\sqrt{2} \in [a,b) \subset [\sqrt{2},2)$.

\hfill $\square$

Problem 16.4.

A map $f : X \to Y$ is said to be an **open map** if for every open set $U$ of $X$, the set $f(U)$ is open in $Y$.

**Proposition.** If $X$ and $Y$ are topological spaces, then the projections $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open maps.

*Proof.* Let $U$ be an open set in $X \times Y$. Then $U$ is a union of basic open sets in $X \times Y$, say $U = \bigcup_\alpha V_\alpha \times W_\alpha$, where each $V_\alpha$ is open in $X$ and each $W_\alpha$ is open on $Y$. Then $\pi_1(U) = \bigcup_\alpha \pi_1(V_\alpha \times W_\alpha) = \bigcup_\alpha V_\alpha$, which is open in $X$, and similarly $\pi_2(U)$ is open in $Y$. \hfill $\square$
Problem 16.9.

**Proposition.** The dictionary order topology on the set $\mathbb{R} \times \mathbb{R}$ is the same as the product topology $\mathbb{R}_d \times \mathbb{R}$, where $\mathbb{R}_d$ denotes $\mathbb{R}$ in the discrete topology.

**Proof.** Because singleton sets are a basis for the topology on $\mathbb{R}_d$, the collection

$$\mathcal{B} = \{\{a\} \times (b, c) \mid a, b, c \in \mathbb{R} \text{ and } b < c\}$$

is a basis for the product topology on $\mathbb{R}_d \times \mathbb{R}$. Each of these sets is open in the dictionary order topology, since $\{a\} \times (b, c) = (a \times b, a \times c)$, and thus the topology on $\mathbb{R}_d \times \mathbb{R}$ is contained in the dictionary order topology.

For the other direction, observe that every ray in the dictionary order topology is open in $\mathbb{R}_d \times \mathbb{R}$. In particular,

$$(a \times b, \infty) = (\{a\} \times (b, \infty)) \cup ((a, \infty) \times \mathbb{R})$$

and

$$(-\infty, a \times b) = ((\infty, a) \times \mathbb{R}) \cup (\{a\} \times (-\infty, b)).$$

Since the rays are a subbasis for the dictionary order topology, it follows that the dictionary order topology is contained in the product topology on $\mathbb{R}_d \times \mathbb{R}$. \hfill \Box

The dictionary order topology on $\mathbb{R} \times \mathbb{R}$ contains the standard topology. In particular, every basic open rectangle $(a, b) \times (c, d)$ in the standard topology is also open in $\mathbb{R}_d \times \mathbb{R}$, since $(a, b)$ is open in $\mathbb{R}_d$ and $(c, d)$ is open in $\mathbb{R}$. This containment of topologies is strict, e.g. $\{0\} \times (0, 1)$ is open in the dictionary order topology but not in the standard topology.