1. Let $\vec{X}(u, v) = (u \cos v, u \sin v, uv)$. Compute the matrix for the corresponding first fundamental form.

2. Let $\vec{X}(u, v)$ be a surface parametrization, and suppose that the corresponding first fundamental form is

$$ g(u, v) = \begin{bmatrix} u + v & \sqrt{v} \\ \sqrt{v} & 1 \end{bmatrix}. $$

(a) Let $\vec{x}(t) = (t, 1)$ be a curve on the $uv$-plane, and let $\vec{y}(t) = \vec{X}(\vec{x}(t))$ be the corresponding curve on the surface. Compute the length of $\vec{y}(t)$ for $0 < t < 3$.

(b) Let $\mathcal{R}$ be the region in the $uv$-plane defined by $0 < u < 4$ and $0 < v < 4$, and let $\vec{X}(\mathcal{R})$ be the corresponding region on the surface. Find the area of $\vec{X}(\mathcal{R})$.

3. Let $C$ be the portion of the cylinder $x^2 + y^2 = 1$ lying above the $xy$-plane, let $P$ be the paraboloid $z = x^2 + y^2$, and let $f: C \to P$ be the map

$$ f(x, y, z) = (x \sqrt{z}, y \sqrt{z}, z). $$

(a) Compute $df_p(0, 0, 1)$ and $df_p(1, 0, 0)$, where $p$ is the point $(0, 1, 4)$.

(b) Compute the Jacobian of $f$ at the point $(0, 1, 4)$.

4. Let $S_1$ be the cylinder $x^2 + y^2 = 1$ for $z > 0$, let $S_2$ be the cone $z = \sqrt{x^2 + y^2}$, and let $f: S_1 \to S_2$ be the map

$$ f(x, y, z) = (xe^{kz}, ye^{kz}, e^{kz}) $$

where $k$ is a constant. Find a value of $k$ for which $f$ is conformal.

5. Let $S$ be the surface $z = \cos(3x) + 6 \sin(xy)$, oriented so that normal vectors point upwards.

(a) Compute the principle curvatures of $S$ at the point $(0, 0, 1)$.

(b) Compute the Gaussian curvature and mean curvature of $S$ at this point.
6. Let \( C \) be a circle of radius 5 that contains the points \((0, 0, 4)\) and \((0, 0, -4)\), and let \( A \) be the (open) minor arc of \( C \) between these points. Let \( S \) be the surface obtained by rotating the arc \( A \) around the \( z \)-axis, oriented so that normal vectors point outwards.

Note that the surface \( S \) does not include the cusp points \((0, 0, 4)\) and \((0, 0, -4)\).

(a) Find the principle curvatures of \( S \) at the point \((2, 0, 0)\).

(b) Find the principle curvatures of \( S \) at the point \((1, 0, 3)\).

(c) Find the image of \( S \) under the Gauss map. Express your answer as one or more inequalities defining a region on the unit sphere.

(d) Use your answer to part (c) to evaluate \( \iint_{S} K \,dA \), where \( K \) is the Gaussian curvature of \( S \).

7. Let \( S \) be the surface \( r = 2 + \cos z \) for \(-\pi < z < \pi\), oriented with normal vectors pointing outwards.

(a) Find the principle curvatures of \( S \) at the point \((3, 0, 0)\).

(b) Find the principle curvatures of \( S \) at the point \((2, 0, \pi/2)\).

(c) For what values of \( z \) in the range \(-\pi < z < \pi\) is the Gaussian curvature of \( S \) positive?

8. Let \( P \) be the paraboloid \( z = \frac{2}{3}(x^2 + y^2) \), oriented so that the normal vectors point inwards, and consider the curve on this surface defined by \( \vec{x}(t) = (\cos t, \sin t, 2/3) \).

(a) Find the vectors \( \{\vec{N}, \vec{T}, \vec{U}\} \) of the Darboux frame for \( \vec{x} \) at the point \((1, 0, 2/3)\).

(b) Find the normal curvature \( \kappa_n \) and geodesic curvature \( \kappa_g \) of \( \vec{x} \) at the point \((1, 0, 2/3)\).

(c) Use the Gauss map to compute \( \iint_{\mathcal{R}} K \,dA \), where \( \mathcal{R} \) is the portion of the surface for which \( z < 2/3 \).