Maps Between Surfaces

Outline

1. Linear Transformations

If $V$ and $W$ are vector spaces, a linear transformation from $V$ to $W$ is a function $T: V \to W$ such that

1. $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$, and
2. $T(\lambda \vec{v}) = \lambda T(\vec{v})$

for all $\vec{v}_1, \vec{v}_2, \vec{v} \in V$ and $\lambda \in \mathbb{R}$. Linear transformations $\mathbb{R}^m \to \mathbb{R}^n$ correspond to $n \times m$ matrices, but linear transformations between other vector spaces don’t correspond to matrices in any natural way.

2. Differentials of Maps

A map between surfaces is a differentiable function $f: S_1 \to S_2$, where $S_1$ and $S_2$ are regular surfaces. (We will write $f$ instead of $\vec{f}$ to help the notation look a little cleaner.) The differential of such a map at a point $p \in S_1$ is a linear transformation

$$df_p: T_p S_1 \to T_{f(p)} S_2.$$

That is, $df_p$ takes as input a tangent vector $\vec{t}$ at $p$, and outputs a corresponding tangent vector $df_p(\vec{t})$ at $f(p)$.

The differential $df_p$ can be defined using curves. Given a tangent vector $\vec{t}$ at $p$, let $\vec{x}(t)$ be a curve on $S_1$ such that $\vec{x}(a) = p$ and $\vec{x}'(a) = \vec{t}$. Then

$$df_p(\vec{t}) = \vec{y}'(a),$$

where $\vec{y}(t) = f(\vec{x}(t))$ is the corresponding curve on $S_2$.

Note: We won’t actually use the definition of a differential very often. In most cases, it is much easier to compute with differentials using a parametrization, as described below.

3. Differentials using Parametrizations

Let $f: S_1 \to S_2$ be a map between surfaces, and let $\vec{X}: U \to S_1$ be a parametrization. Let $\vec{F}: U \to S_2$ be the composition

$$\vec{F}(u,v) = f(\vec{X}(u,v)).$$

In this case, it follows from the chain rule that

$$df_p(\vec{X}_u) = \vec{F}_u \quad \text{and} \quad df_p(\vec{X}_v) = \vec{F}_v$$

for any point $p$. More precisely,

$$df_{\vec{X}(u,v)}(\vec{X}_u(u,v)) = \vec{F}_u(u,v) \quad \text{and} \quad df_{\vec{X}(u,v)}(\vec{X}_v(u,v)) = \vec{F}_v(u,v).$$
4. Types of Maps

Let \( f: S_1 \rightarrow S_2 \) be a map between two surfaces, let \( \tilde{X}: U \rightarrow S_1 \) be a parametrization of \( S_1 \), and let \( \tilde{F}(u, v) = f(\tilde{X}(u, v)) \).

1. We say that \( f \) is equiareal if \( \|\tilde{F}_u \times \tilde{F}_v\| = \|\tilde{X}_u \times \tilde{X}_v\| \).

2. We say that \( f \) is an isometry if it is bijective and
   \[
   \tilde{F}_u \cdot \tilde{F}_u = \tilde{X}_u \cdot \tilde{X}_u, \quad \tilde{F}_u \cdot \tilde{F}_v = \tilde{X}_u \cdot \tilde{X}_v, \quad \text{and} \quad \tilde{F}_v \cdot \tilde{F}_v = \tilde{X}_v \cdot \tilde{X}_v.
   \]

3. We say that \( \tilde{f} \) is conformal if there exists a positive scalar \( \lambda = \lambda(u, v) \) so that
   \[
   \tilde{F}_u \cdot \tilde{F}_u = \lambda \tilde{X}_u \cdot \tilde{X}_u, \quad \tilde{F}_u \cdot \tilde{F}_v = \lambda \tilde{X}_u \cdot \tilde{X}_v, \quad \text{and} \quad \tilde{F}_v \cdot \tilde{F}_v = \lambda \tilde{X}_v \cdot \tilde{X}_v.
   \]

Note that a bijective map is an isometry if and only if it is both equiareal and conformal.

5. The Jacobian

Let \( f: S_1 \rightarrow S_2 \) be a map between two surfaces, let \( \tilde{X}: U \rightarrow S_1 \) be a parametrization of \( S_1 \), and let \( \tilde{F}(u, v) = f(\tilde{X}(u, v)) \). Then the Jacobian of \( f \) at a point \( p \) is defined by the formula
   \[
   Jf(p) = \frac{\|\tilde{F}_u \times \tilde{F}_v\|}{\|\tilde{X}_u \times \tilde{X}_v\|}.
   \]

More precisely,
   \[
   Jf(\tilde{X}(u, v)) = \frac{\|\tilde{F}_u(u, v) \times \tilde{F}_v(u, v)\|}{\|\tilde{X}_u(u, v) \times \tilde{X}_v(u, v)\|}.
   \]

Note that \( f \) is equiareal if and only if \( Jf = 1 \).

The Jacobian measures the area expansion of the map \( f \) around the point \( p \). That is, if \( R_1 \) is an infinitesimal region around \( p \) with area \( dA_1 \), and \( R_2 = f(R_1) \) is the corresponding region in \( S_2 \), then the area \( dA_2 \) of \( R_2 \) is given by the formula
   \[
   dA_2 = Jf(p) dA_1.
   \]

More generally, if \( R_1 \) is any region in \( S_1 \) on which \( f \) is one-to-one, and \( R_2 = f(R_1) \) is the corresponding region in \( S_2 \), then
   \[
   \text{area}(R_2) = \int_{R_1} Jf \ dA.
   \]