Solutions to Exam Practice Problems
Math 352, Fall 2011

1. Note first that \((0,0,1) = \gamma(0)\). Now \(\dot{\gamma}(t) = (2t + 1, \cos t, e^t)\) and \(\ddot{\gamma}(t) = (2, -\sin t, e^t)\), so \(\dot{\gamma}(0) = (1,1,1)\) and \(\ddot{\gamma}(0) = (2,0,1)\). The tangent vector \(t\) should be a unit vector parallel to \(\dot{\gamma}(0)\), so

\[
\mathbf{t} = \frac{1}{\sqrt{3}} (1,1,1)
\]

To find the normal vector \(n\), we must find the component of \(\ddot{\gamma}(0)\) that is perpendicular to \(t\):

\[
\ddot{\gamma}(0) - (\ddot{\gamma}(0) \cdot \mathbf{t}) \mathbf{t} = (2,0,1) - \frac{3}{\sqrt{3}} \left( \frac{1}{\sqrt{3}} (1,1,1) \right) = (1,-1,0).
\]

Then \(n\) is a unit vector in this same direction:

\[
\mathbf{n} = \frac{1}{\sqrt{2}} (1,-1,0)
\]

Finally, the binormal vector \(b\) is equal to \(t \times n\):

\[
\mathbf{b} = \frac{1}{\sqrt{3}} (1,1,1) \times \frac{1}{\sqrt{2}} (1,-1,0) = \frac{1}{\sqrt{6}} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} = \frac{1}{\sqrt{6}} (1,1,-2)
\]

2. The arc length of \(\gamma(t)\) is

\[
s = \int \|\dot{\gamma}(t)\| \, dt
\]

\[
= \int \sqrt{(2e^t \cos t - 2e^t \sin t)^2 + (2e^t \sin t + 2e^t \cos t)^2 + (e^t)^2} \, dt
\]

\[
= \int e^t \sqrt{4 \cos^2 t - 8 \cos t \sin t + 4 \sin^2 t + 4 \sin^2 t + 8 \sin t \cos t + 4 \cos^2 t + 1} \, dt
\]

\[
= \int 3e^t \, dt = 3e^t + C.
\]

In particular, the arc length from \(\gamma(0) = (2,0,1)\) to \(\gamma(\pi) = (-2e^\pi,0,e^\pi)\) is \(3e^\pi - 3\).

For a unit speed parametrization, we solve for \(t\) in terms of \(s\). Assuming \(C = 0\), this gives:

\[
t = \log(s/3)
\]

Substituting this into the formula for \(\gamma(t)\) gives the following unit-speed parametrization:

\[
s \mapsto \left( \frac{2s}{3} \cos \left( \log \frac{s}{3} \right), \frac{2s}{3} \sin \left( \log \frac{s}{3} \right), \frac{s}{3} \right)
\]
3. The turning angle for $\gamma$ is

$$\tau(s) = \int \kappa_s(s) \, ds = \int s^2 \, ds = \frac{1}{3}s^3 + C.$$ 

Since $\dot{\gamma}(0) = (0, 1)$, we can choose $\tau(0) = \pi/2$, in which case $C = \pi/2$. Then

$$\dot{\gamma}(s) \equiv (\cos \tau(s), \sin \tau(s)) = \left(\cos \left(\frac{1}{3}s^3 + \frac{\pi}{2}\right), \sin \left(\frac{1}{3}s^3 + \frac{\pi}{2}\right)\right)$$

4. (a) Note that

$$\dot{\gamma}(t) = (-\sin t - 2\sin 2t, \cos t - 2\cos 2t)$$

and

$$\ddot{\gamma}(t) = (-\cos t - 4\cos 2t, -\sin t + 4\sin 2t).$$

In particular, at the point $(2, 0) = \gamma(0)$, we have $\dot{\gamma}(0) = (0, -1)$ and $\ddot{\gamma}(0) = (-5, 0)$. Since $\ddot{\gamma}(0)$ is clockwise from $\dot{\gamma}(0)$, the signed curvature is negative, so $\kappa_s(0) = -5$.

(b) The osculating circle must be directly to the left of $(2, 0)$, and must have a radius of $1/5$. Therefore, the equation for the circle is

$$\left(\frac{x - 9}{5}\right)^2 + y^2 = \frac{1}{25}$$

(c) We know that the value of this integral is $2\pi n$, where $n$ is the number of counterclockwise rotations made by the unit tangent vector $t$. The following picture shows how the direction of $t$ changes as we travel around the curve:

As you can see, $t$ makes two full clockwise rotations, and therefore the value of the integral is $-4\pi$.

5. By Green’s theorem, $\int_\gamma 2y \, dx + 5x \, dy = \int_{\text{int}(\gamma)} 3 \, dx \, dy = 3A(\gamma)$, where $A(\gamma)$ denotes the area inside $\gamma$. By the Isoperimetric Inequality, this area will be maximized when $\gamma$ is a circle of radius $5/\pi$. In particular, the maximum possible area is $\pi(5/\pi)^2 = 25/\pi$, so the maximum possible value of the integral is $\frac{75}{\pi}$. 
6. Since $\|\gamma(t)\| = 1$, we know that $\gamma(t) \cdot \gamma(t) = 1$. Taking the derivative of this equation gives
\[
\dot{\gamma}(t) \cdot \gamma(t) + \gamma(t) \cdot \dot{\gamma}(t) = 0,
\]
which simplifies to
\[
\gamma(t) \cdot \dot{\gamma}(t) = 0.
\]
Taking the derivatives again yields
\[
\dot{\gamma}(t) \cdot \dot{\gamma}(t) + \gamma(t) \cdot \ddot{\gamma}(t) = 0.
\]
Since $\gamma$ is unit speed, we know that $\dot{\gamma}(t) \cdot \dot{\gamma}(t) = \|\dot{\gamma}(t)\|^2 = 1^2 = 1$, and therefore $\gamma(t) \cdot \dot{\gamma}(t) = -1$.

7. As the small circle rolls, the arc length rolled along the small circle is the same as the arc length rolled along the big circle. This gives us the following picture:

![Diagram](image)

The large circle has circumference $32\pi$, so the highlighted arc on the large circle has a length of $8\pi$. The highlighted arc on the small circle must have the same length. Since the total circumference of the small circle is $12\pi$, it follows that the length of the arc from $P$ to $Q$ is $\pi$, so the angular distance from $P$ to $Q$ is $\pi/6$ radians. Then the position of $P$ at the end is
\[
(10, 0) + 6(\cos \pi/6, \sin \pi/6) = (10 + 3\sqrt{3}, 3).
\]

8. Since the curve lies on the graph $z = f(x, y)$, we know that
\[
\gamma_3(t) = f(\gamma_1(t), \gamma_2(t)).
\]
Taking the derivative of this equation using the Chain Rule gives
\[
\dot{\gamma}_3(t) = f_x(\gamma_1(t), \gamma_2(t)) \dot{\gamma}_1(t) + f_y(\gamma_1(t), \gamma_2(t)) \dot{\gamma}_2(t)
\]
so
\[
\dot{\gamma}_3(0) = (9)(2) + (4)(-3) = 6.
\]
We conclude that $\dot{\gamma}(0) = (2, -3, 6)$, so $\mathbf{t} = \left( \frac{2}{7}, -\frac{3}{7}, \frac{6}{7} \right)$. 
9. Since $\gamma$ has torsion 0, the curve $\gamma$ must lie entirely in a single plane. Since the curvature is 1, it must be a circle with radius 1. The values of $\gamma(0)$, $\dot{\gamma}(0)$, and $\ddot{\gamma}(0)$ tell us exactly where this circle is:

Then $\gamma(\pi/2)$ is a quarter-turn around this circle, at the point $(3, 0, 1)$.

10. The center of the circle is halfway between $(3, 3, 4)$ and $(5, 7, 8)$, at the point

$$\frac{1}{2} (3, 3, 4) + \frac{1}{2} (5, 7, 8) = (4, 5, 6).$$

Let $\mathbf{v}$ be the vector from the center to the point on the right of the circle, and let $\mathbf{w}$ be the vector from the center to the point on the top of the circle. Then the desired parametrization is

$$\gamma(t) = (4, 5, 6) + (\cos t)\mathbf{v} + (\sin t)\mathbf{w}$$

$$= (4, 5, 6) + (\cos t)(1, 2, 2) + (\sin t)(-2, 2, -1)$$

$$= \begin{pmatrix} 4 \cos t - 2 \sin t, & 5 + 2 \cos t + 2 \sin t, & 6 + 2 \cos t - \sin t \end{pmatrix}$$