1. Let $\sigma: (0, 2) \times (0, 2\pi) \to S$ be the surface patch 

$$\sigma(r, \theta) = (r \cos \theta, r \sin \theta, r^2).$$

Then the image of $\sigma$ is almost all of $S$, and the Jacobian of $\sigma$ is

$$\|\sigma_r \times \sigma_\theta\| = \|\sigma_r\| \|\sigma_\theta\| = \|\cos \theta, \sin \theta, 2r\| \|(-r \sin \theta, r \cos \theta, 0)\| = r\sqrt{1 + 4r^2}.$$

(a) The surface area is 

$$\int_0^{2\pi} \int_0^2 r\sqrt{1 + 4r^2} \, dr \, d\theta = \frac{(17\sqrt{17} - 1)\pi}{6}.$$

(b) 

$$\int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2} r\sqrt{1 + 4r^2} \, dr \, d\theta = 2\pi \int_0^2 r(1 + 4r^2) \, dr = 36\pi.$$

2. Let $F(x, y, z) = x^2 + 2y^2 + 3z^2$ be the obvious extension of $f$ to $\mathbb{R}^3$. We wish to solve the equations 

$$5x + 4y + 3z = 36 \quad \text{and} \quad \nabla F(x, y, z) = \lambda(5, 4, 3),$$

where $(5, 4, 3)$ is the normal vector to the plane. Since $\nabla F = (2x, 4y, 6z)$, we get the following four equations:

$$5x + 4y + 3z = 36, \quad 2x = 5\lambda, \quad 4y = 4\lambda, \quad \text{and} \quad 6z = 3\lambda.$$ 

Solving the latter three equations for $x, y$ and $z$ and plugging them into the first equation, we get $18\lambda = 36$, so $\lambda = 2$. Plugging this back into the latter three equations gives the point $(5, 2, 1)$.

3. We have $\sigma_u = (\cos v, \sin v, v)$ and $\sigma_v = (-u \sin v, u \cos v, u)$, so

$$I = \begin{bmatrix} \sigma_u \cdot \sigma_u & \sigma_u \cdot \sigma_v \\ \sigma_v \cdot \sigma_u & \sigma_v \cdot \sigma_v \end{bmatrix} = \begin{bmatrix} 1 + v^2 & uv \\ uv & 2u^2 \end{bmatrix}.$$

4. (a) This is the same as the angle between the radial vectors $p$ and $q$. Both are these are unit vectors, so the angle is

$$\cos^{-1}(p \cdot q) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}.$$
(b) The small circle is up at an angle of $\pi/4$, so the Euclidean radius of the small circle is $\sqrt{2}/2$, and therefore the circumference is $\pi\sqrt{2}$. The two points are a quarter-circle apart, and therefore the length of the arc is $\frac{\pi\sqrt{2}}{4}$.

5. The curve is $u = t$ and $v = t$, so $\dot{u} = \dot{v} = 1$. Therefore, the length is
\[
\int_{\gamma} \sqrt{(u^2) \dot{u}^2 + (2uv) \dot{u} \dot{v} + (v^2) \dot{v}^2} \, dt = \int_{0}^{1} \sqrt{t^2 + 2t^2 + t^2} \, dt = \int_{0}^{1} 2t \, dt = 1.
\]

6. Let $t_1 = (-y, x, 0)$ and $t_2 = (0, 0, 1)$. Then $Df(t_1) = 2t_1$ and $Df(t_2) = kt_2$.

(a) The map $f$ is conformal if and only if $Df(t_1)$ and $Df(t_2)$ are orthogonal and have the same length. This occurs if and only if $k = \pm 2$.

(b) The Jacobian of $f$ is $\|Df(t_1) \times Df(t_2)\| = |2k|$. Thus $f$ is equiareal if and only if $k = \pm 1/2$.

7. (a) The horizontal geodesic through the point $(2, 0, 0)$ is a circle of radius 2, and the vertical geodesic through this point is an arc of a circle of radius 5. Therefore, the principal curvatures are $\frac{1}{2}$ and $\frac{1}{5}$.

(b) The Gaussian curvature is $\frac{1}{10}$.

(c) The image is the region $-4/5 < z < 4/5$ on the unit sphere.

(d) We must find the area of the image of the Gauss map. Using Archimedes’ map, the projection of the image is a cylinder of height $8/5$ and circumference $2\pi$, so the area of the image is $\frac{16\pi}{5}$.

8. (a) Clearly $N = (3, 0, -1)/\sqrt{10}$ and $t = (0, 1, 0)$. Taking the cross product yields $g = (1, 0, 3)/\sqrt{10}$.

(b) Observe that $\ddot{y} = (-1, 0, 0)$. Then $\kappa_n = \ddot{y} \cdot N = \frac{3}{\sqrt{10}}$ and $\kappa_g = \ddot{y} \cdot g = -\frac{1}{\sqrt{10}}$.

9. The eigenvalues of the matrix $\begin{bmatrix} 7 & 2 \\ 2 & 7 \end{bmatrix}$ are 9 and 5, so the eigenvalues of II are $-\frac{2}{27}(9) = -\frac{2}{3}$ and $-\frac{2}{27}(5) = -\frac{10}{27}$. The principal curvatures of $S$ are the negatives of these, namely $\frac{2}{3}$ and $\frac{10}{27}$. 
10. \[
\begin{array}{c|c|c|c}
 & T_A & T_B & T_C \\
\hline
\text{Orientation-Preserving} & X & X & \text{ } \\
\hline
\text{Conformal} & & X & X \\
\hline
\text{Equiareal} & X & X & \text{ } \\
\hline
\text{Isometric} & \text{ } & X & \text{ } \\
\end{array}
\]

11. Let \( \sigma: (0, 1) \times (0, 3) \to \mathcal{S} \) be the surface patch \( \sigma(x, y) = (x, y, x^2) \). Then \( \sigma_x = (1, 0, 2x) \) and \( \sigma_y = (0, 1, 0) \) so the Jacobian of \( \sigma \) is \( \sqrt{1 + 4x^2} \). Then the integral is
\[
\int_0^1 \int_0^3 xy \sqrt{1 + 4x^2} \, dy \, dx = \int_0^1 x \sqrt{1 + 4x^2} \, dx \int_0^3 y \, dy = \frac{-3 + 15\sqrt{5}}{8}
\]

12. (a) We have \( \mathbf{N} = (0, \cos t, \sin t) \) and \( \mathbf{t} = (0, -\sin t, \cos t) \). Taking the cross product yields \( \mathbf{g} = (1, 0, 0) \).

(b) Since \( \mathbf{t}(0) = (0, 0, 1) \) and \( \mathbf{g}(0) = (1, 0, 0) \), we see that \( \mathbf{v}(0) = 3\mathbf{t}(0) + \mathbf{g}(0) \).
Since \( \gamma \) is a geodesic, it follows that
\[
\mathbf{v}(t) = 3\mathbf{t} + \mathbf{g} = (1, -3\sin t, 3\cos t)
\]