1. Let $S$ be the portion of the paraboloid $z = x^2 + y^2$ lying below the plane $z = 4$.
   (a) Find the surface area of $S$.
   (b) Evaluate $\int \int_S \sqrt{1 + 4z} \, dA$.

2. Let $\mathcal{P}$ be the plane $5x + 4y + 3z = 36$, and let $f : \mathcal{P} \to \mathbb{R}$ be the function
   $$f(x, y, z) = x^2 + 2y^2 + 3z^2.$$ 
   Use the method of Lagrange multipliers to find the critical point for $f$ on $\mathcal{P}$.

3. Let $\sigma : (0, \infty) \times \mathbb{R} \to S$ be the surface patch
   $$\sigma(u, v) = (u \cos v, u \sin v, uv).$$
   Compute the matrix for the first fundamental form of $S$ with respect to $\{\sigma_u, \sigma_v\}$.

4. Let $S^2$ be the unit sphere, and let
   $$p = \left( \frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2} \right) \quad \text{and} \quad q = \left( -\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2} \right).$$
   (a) Find the length of the arc of the great circle from $p$ to $q$.
   (b) Find the length of the arc of the small circle $z = \sqrt{2}/2$ from $p$ to $q$.

5. Let $S$ be a smooth surface, let $\sigma : \mathbb{R} \times (0, \infty) \to S$ be a surface patch, and suppose that the first fundamental form of $S$ is
   $$I = \begin{bmatrix} u^2 & uv \\ uv & v^2 \end{bmatrix}.$$ 
   Use this information to find the length of the curve $\gamma(t) = \sigma(t, t)$ for $0 \leq t \leq 1$.

6. Let $\mathcal{C}_1$ be the cylinder $x^2 + y^2 = 1$, let $\mathcal{C}_2$ be the cylinder $x^2 + y^2 = 4$, and let $f : \mathcal{C}_1 \to \mathcal{C}_2$ be the map
   $$f(x, y, z) = (2x, 2y, kz),$$ 
   where $k$ is a constant.
   (a) For what real values of $k$ is the map $f$ conformal? Explain.
   (b) For what real values of $k$ is the map $f$ equiareal? Explain.
7. Let \( p \) and \( q \) be the points \((0,0,-4)\) and \((0,0,4)\). Let \( C \) be a circle of radius 5 that contains \( p \) and \( q \), and let \( A \) be the (open) arc of \( C \) from \( p \) to \( q \). Let \( S \) be the surface obtained by rotating the arc \( A \) around the \( z \)-axis.

\[ \text{Note that the surface } S \text{ does not include the cusp points } p \text{ and } q. \]

(a) Find the principal curvatures of \( S \) at the point \((2,0,0)\).

(b) Find the Gaussian curvature of \( S \) at the point \((2,0,0)\).

(c) Find the image of \( S \) under the Gauss map.

(d) Use your answer to part (c) to evaluate \( \iint_S K \, dA \), where \( K \) is the Gaussian curvature of \( S \).

8. Let \( C \) be the cone \( z = 3r \) with outward-pointing normal vectors, and let \( \gamma : \mathbb{R} \to C \) be the curve \( \gamma(t) = (\cos t, \sin t, 3) \).

(a) Compute the vectors \( \{N, t, g\} \) of the Darboux frame for \( \gamma \) at the point \((1,0,3)\).

(b) Compute the normal and geodesic curvatures of \( \gamma \) at the point \((1,0,3)\).

9. Let \( S \) be the ellipsoid \( x^2 + 2y^2 + 2z^2 = 5 \), and let \( p = (1,1,1) \). Then the vectors

\[
\mathbf{t}_1 = \frac{1}{3}(2,1,-2) \quad \text{and} \quad \mathbf{t}_2 = \frac{1}{3}(-2,2,-1)
\]

are a basis for \( T_pS \), and the second fundamental form for \( S \) at \( p \) with respect to this basis is

\[
\mathbb{II} = -\frac{2}{27} \begin{bmatrix} 7 & 2 \\ 2 & 7 \end{bmatrix}.
\]

Use this information to find the principal curvatures of \( S \) at the point \( p \).
10. Let $T_A$, $T_B$, $T_C$, and $T_D$ be the linear transformations of $\mathbb{R}^2$ represented by the following matrices:

$$
A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 \\ 2 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix}.
$$

For each entry of the following table, place an “X” if the given linear transformation has the given property:

<table>
<thead>
<tr>
<th></th>
<th>$T_A$</th>
<th>$T_B$</th>
<th>$T_C$</th>
<th>$T_D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Orientation-Preserving</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Conformal</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Equiareal</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Isometric</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

11. Evaluate $\int \int_S xy \, dA$, where $S$ is the portion of the surface $z = x^2$ for which $0 \leq x \leq 1$ and $0 \leq y \leq 3$.

12. Let $\gamma : [0, \pi/2] \rightarrow S^2$ be the curve $\gamma(t) = (0, \cos t, \sin t)$.

(a) Find the vectors $\{N, t, g\}$ of the Darboux frame for $\gamma$ at time $t$.

(b) Let $v(t)$ be a parallel vector field along $\gamma$, and suppose that $v(0) = (1, 0, 3)$. Find a formula for $v(t)$.