1. Let $S^2$ be the unit sphere in $\mathbb{R}^3$. Given a point $(u, v) \in \mathbb{R}^2$, let $L$ be the line through $(u, v, 0)$ and $(0, 0, 1)$ in $\mathbb{R}^3$, and let $\sigma(u, v)$ be the point in the region $z < 1$ at which $L$ intersects $S^2$.

(a) Find a formula for $\sigma(u, v)$ in terms of $u$ and $v$. \textit{(Hint: Start by parameterizing the line $L$, and then solve for the points of intersection.)}

(b) What is the image of the surface patch $\sigma : \mathbb{R}^2 \to S^2$? Explain how $\sigma$ can be used to construct an atlas for $S^2$ with only two patches.

(c) Find the tangent vectors $\sigma_u$ and $\sigma_v$ at the point $(1/3, 2/3, 2/3)$.

2. Let $C$ be the cylinder $x^2 + y^2 = 1$ in $\mathbb{R}^3$. Find an open set $U \subset \mathbb{R}^2$ and a surface patch $\sigma : U \to C$ such that $\sigma(U) = C$.

3. Let $S^2$ be the unit sphere in $\mathbb{R}^3$, let $U$ be the open unit disk in $\mathbb{R}^2$, and let $\sigma : U \to S^2$ and $\tilde{\sigma} : U \to S^2$ be the following surface patches:

$$\sigma(u, v) = (u, v, \sqrt{1 - u^2 - v^2}) \quad \text{and} \quad \tilde{\sigma}(\bar{u}, \bar{v}) = (\bar{u}, \sqrt{1 - \bar{u}^2 - \bar{v}^2}, \bar{v}).$$

(a) Let $\Phi$ be the transition map from $\sigma$ to $\tilde{\sigma}$. Determine the domain and range of $\Phi$, and find a formula for $\Phi(u, v)$.

(b) Compute $D_p \Phi$ at the point $p = (0.48, 0.36)$.

(c) Make a drawing showing the domain and range of $\Phi$. Include several horizontal and vertical gridlines in the domain, and sketch the corresponding parameter curves in the range. Also, draw the vectors $(1, 0)$ and $(0, 1)$ emanating from the point $p = (0.48, 0.36)$ in the domain, and show the corresponding vectors $\Phi_u(p)$ and $\Phi_v(p)$ at the point $\Phi(p)$ in the range.