Curves on Surfaces

Outline

1. The Darboux Frame
   Let $S$ be a surface, and let $\gamma$ be a curve on $S$. At each point on $\gamma$, consider the following three vectors:
   - The unit normal vector $\mathbf{N}$ to the surface.
   - The unit tangent vector $\mathbf{t}$ to the curve $\gamma$.
   - The tangent normal vector $\mathbf{g} = \mathbf{N} \times \mathbf{t}$. This vector is tangent to the surface $S$, but normal to the curve $\gamma$.
   These vectors $\{\mathbf{N}, \mathbf{t}, \mathbf{g}\}$ form a right-handed frame, known as the **Darboux frame** for $\gamma$ on $S$.
   If we position the surface so that the normal vector $\mathbf{N}$ points towards us, then the tangent normal vector $\mathbf{g}$ points $90^\circ$ counterclockwise from the tangent vector $\mathbf{t}$.

2. Normal and Geodesic Curvature
   Now suppose that $\gamma$ is a unit-speed curve. Then $\ddot{\mathbf{y}}$ is perpendicular to $\mathbf{t}$, but $\ddot{\mathbf{y}}$ may have components in the normal and tangent normal directions:
   \[
   \ddot{\mathbf{y}} = \kappa_n \mathbf{N} + \kappa_g \mathbf{g}.
   \]
   The quantity $\kappa_n$ is called the **normal curvature** of $\gamma$, and $\kappa_g$ is called the **geodesic curvature**. These are related to the total curvature $\kappa$ of $\gamma$ by the formula
   \[
   \kappa^2 = \|\ddot{\mathbf{y}}\|^2 = \kappa_n^2 + \kappa_g^2.
   \]
   Note that $\gamma$ is a geodesic if and only if $\kappa_g = 0$, in which case the normal curvature is the same as the curvature. In general, the geodesic curvature $\kappa_g$ measures the extent to which $\gamma$ fails to be a geodesic.
   Both $\kappa_n$ and $\kappa_g$ may be positive or negative. Specifically, $\kappa_n$ is positive if $\gamma$ curves towards the normal vector $\mathbf{N}$, and $\kappa_g$ is positive if $\gamma$ curves towards the tangent normal vector $\mathbf{g}$.

3. Frenet-Serret Formulas for the Darboux Frame
   The Darboux frame has its own set of Frenet-Serret formulas. Given a unit-speed curve $\gamma$ on a surface $S$, the unit tangent, unit normal, and tangent normal vectors obey the formulas
   \[
   \begin{align*}
   \dot{\mathbf{N}} &= 0 \mathbf{N} - \kappa_n \mathbf{t} - \tau_g \mathbf{g} \\
   \dot{\mathbf{t}} &= \kappa_n \mathbf{N} + 0 \mathbf{t} + \kappa_g \mathbf{g} \\
   \dot{\mathbf{g}} &= \tau_g \mathbf{N} - \kappa_g \mathbf{t} + 0 \mathbf{g}
   \end{align*}
   \]
   Here $\tau_g$ is something called the “geodesic torsion”, which we shall not concern ourselves with. From our point of view, these equations interesting primarily because they give us some new formulas for $\kappa_n$ and $\kappa_g$:
   \[
   \kappa_n = -\dot{\mathbf{N}} \cdot \mathbf{t} \quad \text{and} \quad \kappa_g = -\dot{\mathbf{g}} \cdot \mathbf{t}
   \]
The first of these is particularly important and useful. Conceptually, it says that the normal curvature of a curve depends only on the shape of the surface and the direction that the curve is traveling.

By the way, the scalar quantity $\tau_g$ that appears in the derivative formulas for $N$ and $g$ is called the “geodesic torsion” of $\gamma$. It plays essentially no role in the theory.

4. The Gauss-Bonnet Theorem for Closed Curves

There is a version of the Gauss-Bonnet theorem for closed curves on a surface:

**Theorem.** Let $S$ be a surface. Let $R$ be a simply-connected region in $S$, and let $C$ be its boundary curve, oriented counterclockwise. Then

$$2\pi - \int_C \kappa_g ds = \iint_R K dA,$$

where $\kappa_g$ is the geodesic curvature of $C$, and $K$ is the Gaussian curvature of $S$.

For example, if $R$ is a simply-connected region on the unit sphere, and $C$ is the boundary curve of $R$, then

$$\text{area of } R = 2\pi - \int_C \kappa_g ds.$$
Practice Problems

1. Let $\gamma$ be a unit-speed curve on $S^2$, and suppose that

$$\gamma(0) = (1, 0, 0), \quad \dot{\gamma}(0) = (0, 4/5, 3/5) \quad \text{and} \quad \ddot{\gamma}(0) = (-1, -6, 8).$$

(a) Compute the Darboux frame $\{N, t, g\}$ for $\gamma$ at $t = 0$.
(b) Find the normal and geodesic curvatures of $\gamma$ at $t = 0$.

2. Let $\gamma$ be a unit-speed curve on the cylinder $x^2 + y^2 = 1$, and suppose that

$$\gamma(0) = (1, 0, 0) \quad \text{and} \quad \dot{\gamma}(0) = (0, 4/5, 3/5).$$

(a) Compute the value of $\dot{N}$ at $t = 0$, where $N$ is the outward-pointing unit normal vector.
(b) Compute the normal curvature of $\gamma$ at $t = 0$.

3. Let $P$ be the paraboloid $z = x^2 + y^2$, and let $C$ be the circle on $P$ obtained by intersecting with the plane $z = 1$.

(a) Compute the normal curvature of $C$.
(b) Find the magnitude of the geodesic curvature of $C$. 
Solutions

1. (a) The normal vector \( \mathbf{N} \) is just \( (1, 0, 0) \), and the tangent vector \( \mathbf{t} \) is \( (0, 4/5, 3/5) \). The tangent normal vector is the cross product of these:
\[
\mathbf{g} = (1, 0, 0) \times (0, 4/5, 3/5) = (0, -3/5, 4/5)
\]
(b) We have \( \ddot{\gamma} = (-1, -6, 8) = -\mathbf{N} + 10\mathbf{g} \), so \( \kappa_n = -1 \) and \( \kappa_g = 10 \). The same answers could also be obtained by taking the dot products \( \ddot{\gamma} \cdot \mathbf{N} \) and \( \ddot{\gamma} \cdot \mathbf{g} \).

2. (a) Since \( \mathbf{N} = (x, y, 0) \), we have \( \dot{\mathbf{N}} = (\dot{x}, \dot{y}, 0) = (0, 4/5, 0) \).
(b) We have \( \kappa_n = -\dot{\mathbf{N}} \cdot \mathbf{t} = -(0, 4/5, 0) \cdot (0, 4/5, 3/5) = -16/25 \).

3. (a) By symmetry, the normal curvature should be the same on all of \( C \), so we will compute the normal curvature at the point \( (1, 0, 1) \). If \( \ddot{\gamma} \) is a unit-speed parametrization, then \( \ddot{\gamma} = (-1, 0, 0) \) at this point, and \( \mathbf{N} = (2, 0, -1)/\sqrt{5} \), so \( \kappa_n = \ddot{\gamma} \cdot \mathbf{N} = -2/\sqrt{5} \).
(b) Since \( \kappa = 1 \) and \( \kappa_n^2 + \kappa_g^2 = \kappa^2 \), the magnitude of \( \kappa_g \) must be \( 1/\sqrt{5} \).