The First Fundamental Form

Outline

1. Bilinear Forms

Let $V$ be a vector space. A bilinear form on $V$ is a function $V \times V \to \mathbb{R}$ that is linear in each variable separately. That is, $\langle -, - \rangle$ is bilinear if

$$\langle \alpha u + \beta v, w \rangle = \alpha \langle u, w \rangle + \beta \langle v, w \rangle \quad \text{and} \quad \langle u, \alpha v + \beta w \rangle = \alpha \langle u, v \rangle + \beta \langle u, w \rangle$$

for all $u, v, w \in V$ and $\alpha, \beta \in \mathbb{R}$.

Any bilinear form on $\mathbb{R}^n$ can be written as

$$\langle v, w \rangle = \sum_{i,j=1}^{n} P_{ij} v_i w_j = P_{11} v_1 w_1 + P_{12} v_1 w_2 + \cdots + P_{nn} v_n w_n.$$

This equation can be written as

$$\langle v, w \rangle = v^T P w,$$

where $P$ is an $n \times n$ matrix of coefficients. The entries of $P$ can be obtained by applying the bilinear form to the basis vectors:

$$P = \begin{bmatrix}
\langle e_1, e_1 \rangle & \langle e_1, e_2 \rangle & \cdots & \langle e_1, e_n \rangle \\
\langle e_2, e_1 \rangle & \ddots & \cdots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
\langle e_n, e_1 \rangle & \cdots & \cdots & \langle e_n, e_n \rangle
\end{bmatrix}.$$

In a similar way, we can define the matrix for any bilinear form on a vector space with respect to a given basis.

2. Inner Products

Let $V$ be a vector space. An inner product on $V$ is a bilinear form $\langle -, - \rangle$ with the following properties:

1. **Positive Definite:** $\langle v, v \rangle > 0$ for all nonzero $v \in V$.
2. **Symmetric:** $\langle v, w \rangle = \langle w, v \rangle$ for all $v, w \in V$.

For example, the dot product $\langle v, w \rangle = v_1 w_1 + \cdots + v_n w_n$ is an inner product on $\mathbb{R}^n$. More generally, a bilinear form

$$\langle v, w \rangle = v^T P w$$

is an inner product if and only if the matrix $P$ is symmetric and positive definite (see below).

A vector space $V$ together with an inner product $\langle -, - \rangle$ on $V$ is called an inner product space. Every inner product space has an associated norm, defined by the formula

$$\|v\| = \sqrt{\langle v, v \rangle}.$$

Thus we can talk about the lengths of vectors in an inner product space. We can also talk about the angle $\theta$ between nonzero vectors, defined by the formula

$$\langle v, w \rangle = \|v\| \|w\| \cos \theta.$$
3. Positive Definite Matrices
A symmetric $n \times n$ matrix $P$ is **positive definite** if it satisfies either of the following equivalent conditions:

- All the eigenvalues of $P$ are positive.
- $v^T P v > 0$ for every nonzero vector $v \in \mathbb{R}^n$.

Positive definite matrices can be used to define inner products on $\mathbb{R}^n$ (see above).

A $2 \times 2$ symmetric matrix is positive definite if and only if its trace and determinant are both positive. For a larger matrix, the following test can be used:

**Sylvester’s Criterion.** *A symmetric matrix $P$ is positive definite if and only if all of the leading principal minors of $P$ are positive.*

Here the phrase **leading principal minor** refers to the determinant of any square submatrix lying in the upper-left corner of $P$. An $n \times n$ matrix has $n$ leading principal minors, the first of which is the upper-left entry, and the last of which is the determinant of the entire matrix.

4. The First Fundamental Form
Let $S$ be a smooth surface, and let $p \in S$. The **first fundamental form** of $S$ at $p$ is the inner product $\langle -, - \rangle_p$ on $T_p S$ obtained by restricting the dot product on $\mathbb{R}^3$ to pairs of tangent vectors.

Given a surface patch $\sigma: U \rightarrow S$, the matrix for the first fundamental form with respect to the basis $\{ \sigma_u, \sigma_v \}$ is

$$
\begin{bmatrix}
E & F \\
F & G
\end{bmatrix}
$$

where $E = \sigma_u \cdot \sigma_u$, $F = \sigma_u \cdot \sigma_v$, and $G = \sigma_v \cdot \sigma_v$.

Given any pair of tangent vectors $v$ and $w$ at the same point satisfying

$$
v = v_1 \sigma_u + v_2 \sigma_v \quad \text{and} \quad w = w_1 \sigma_u + w_2 \sigma_v
$$

The dot product of $v$ and $w$ is given by the formula

$$
v \cdot w = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.
$$

Using the first fundamental form, the infinitesimal distance $ds$ between two nearby points $\sigma(u, v)$ and $\sigma(u + du, v + dv)$ on the surface satisfies the equation

$$ds^2 = Edu^2 + 2F du dv + G dv^2.$$

The expression on the right is classically referred to as the **first fundamental form**. Its integral gives the arc length of a path $\sigma(u(t), v(t))$ on the surface:

$$s = \int \sqrt{E \dddot{u}^2 + 2F \dddot{u} \dddot{v} + G \dddot{v}^2} \, dt.$$
Practice Problems

1. Let $\mathbb{V}$ be a vector space, and let $v, w \in \mathbb{V}$. Let $\langle -, - \rangle$ be a bilinear form on $\mathbb{V}$, and suppose that

$$
\langle v, v \rangle = 2, \quad \langle v, w \rangle = 4, \quad \langle w, v \rangle = 3, \quad \text{and} \quad \langle w, w \rangle = 5.
$$

Compute $\langle v + w, 2v - w \rangle$.

2. Determine which of the following matrices are positive definite.

$$
\begin{bmatrix}
3 & -4 \\
-4 & 6
\end{bmatrix}, \quad
\begin{bmatrix}
2 & 3 \\
3 & 4
\end{bmatrix}, \quad
\begin{bmatrix}
7 & 4 & 8 \\
4 & 2 & 4 \\
8 & 4 & 5
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 2 & 0 \\
2 & 5 & 1 \\
0 & 1 & 3
\end{bmatrix}
$$

3. Let $\langle -, - \rangle$ be the inner product on $\mathbb{R}^2$ corresponding to the matrix $\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$.

(a) Find the length of the vector $(1, 5)$ with respect to this inner product.
(b) Find the angle between the vectors $(1, 0)$ and $(1, 5)$ with respect to this inner product.
(c) Find a nonzero vector orthogonal to $(1, 0)$ with respect to this inner product.

4. Let $\langle -, - \rangle$ be an inner product on $\mathbb{R}^2$, and suppose that

$$
\langle (4, 0), (1, 0) \rangle = 12, \quad \langle (0, 1), (0, 1) \rangle = 6, \quad \text{and} \quad \langle (0, 1), (1, 2) \rangle = 14.
$$

Find the matrix for $\langle -, - \rangle$.

5. Let $\mathcal{S}$ be a smooth surface, let $\sigma: (-2, 2) \times \mathbb{R} \to \mathcal{S}$ be a surface patch, and suppose that the first fundamental form of $\mathcal{S}$ is

$$
ds^2 = du^2 + udu dv + dv^2.
$$

Let $\gamma: [0, 1] \to \mathcal{S}$ be the curve $\gamma(t) = \sigma(t, t)$.

(a) Find the speed of $\gamma$ at $t = 0$ and at $t = 1$.
(b) Find the length of $\gamma$ from $t = 0$ to $t = 1$.
(c) Let $\tilde{\gamma}: [0, 1] \to \mathcal{S}$ be the parameter curve $\tilde{\gamma}(t) = \sigma(t, 1)$. Find the angle between $\gamma$ and $\tilde{\gamma}$ at the point $\sigma(1, 1)$.

6. Let $\sigma: \mathbb{R}^2 \to \mathcal{S}$ be the surface patch $\sigma(u, v) = (u, v, \frac{1}{2}(u^2 - v^2))$.

(a) Compute the first fundamental form of $\mathcal{S}$ with respect to the basis $\{\sigma_u, \sigma_v\}$.
(b) Let $\gamma: [0, 1] \to \mathcal{S}$ be the curve $\gamma(t) = \sigma(\sinh t, \sinh t)$. Use the first fundamental form to compute the length of $\gamma$. 

Solutions

1. Using bilinearity,
\[
\langle v + w, 2v - w \rangle = \langle v, 2v - w \rangle + \langle w, 2v - w \rangle \\
= 2\langle v, v \rangle - \langle v, w \rangle + 2\langle w, v \rangle - \langle w, w \rangle \\
= 2(2) - (4) + 2(3) - (5) = 1
\]

2. (a) The first matrix is positive definite, since the diagonal entries and the determinant are both positive.
(b) The second matrix is not positive definite, since the determinant is −1.
(c) The third matrix is not positive definite, since
\[
\begin{vmatrix}
7 & 4 \\
4 & 2
\end{vmatrix} < 0.
\]
(d) The fourth matrix is positive definite, since \(1 > 0\) and
\[
\begin{vmatrix}
1 & 2 & 0 \\
2 & 5 & 1 \\
0 & 1 & 3
\end{vmatrix} > 0.
\]

3. Let \(v = (1, 5)\), let \(w = (1, 0)\), and let \(M\) be the given matrix.
(a) We have \(\|v\| = \sqrt{\langle v, v \rangle} = \sqrt{v^T M v} = \sqrt{160} = 4\sqrt{10}\).
(b) Since \(\|w\| = \sqrt{5}\) and \(\langle v, w \rangle = 20\), we have
\[
\text{(20)} = (4\sqrt{10})(\sqrt{5})\cos \theta.
\]
Solving gives \(\cos \theta = 1/\sqrt{2}\), so \(\theta = 45^\circ\).
(c) Let \(u = (u_1, u_2)\) be the desired vector. Then \(u^T M w = 0\), which reduces to the equation \(5u_1 + 3u_2 = 0\). Hence \(u = (3, -5)\) is one such vector.

4. \(e_1 = (1, 0)\) and \(e_2 = (0, 1)\). The first equation tells us that \(\langle e_1, e_1 \rangle = 3\), and the second says that \(\langle e_2, e_2 \rangle = 6\). The third can be written \(\langle e_2, e_1 \rangle + 2\langle e_2, e_2 \rangle = 14\), and it follows that \(\langle e_1, e_2 \rangle = 2\). Thus, the matrix is \[
\begin{bmatrix}
3 & 2 \\
2 & 6
\end{bmatrix}.
\]

5. This is the curve \((u, v) = (t, t)\), so \(\dot{u} = \dot{v} = 1\), and therefore
\[
\|\dot{\gamma}\| = \sqrt{\dot{u}^2 + u \dot{u} \dot{v} + \dot{v}^2} = \sqrt{(1)^2 + (t)(1)(1) + (1)^2} = \sqrt{t + 2}.
\]
(a) Using the formula above, \(\|\dot{\gamma}(0)\| = \sqrt{2}\) and \(\|\dot{\gamma}(1)\| = \sqrt{3}\).
(b) The length is \(\int_0^1 \|\dot{\gamma}\| \, dt = \int_0^1 \sqrt{t + 2} \, dt = \frac{6\sqrt{3} - 4\sqrt{2}}{3}\).
(c) The matrix for the first fundamental form at the point $\sigma(1, 1)$ is

$$I = \begin{bmatrix} 1 & u/2 \\ u/2 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1 \end{bmatrix}$$

The two tangent vectors are $(1, 1)$ and $(1, 0)$. We compute the norms and inner product:

$$\|\sigma(1, 1)\| = \sqrt{3}, \quad \|\sigma(1, 0)\| = 1, \quad \text{and} \quad \langle (1, 1), (1, 0) \rangle = 3/2.$$

This gives us $(3/2) = (\sqrt{3})(1)\cos \theta$. Then $\cos \theta = \sqrt{3}/2$, so $\theta = 30^\circ$.

6. (a) We have $\sigma_u = (1, 0, u)$ and $\sigma_v = (0, 1, -v)$, so

$$I = \begin{bmatrix} \sigma_u \cdot \sigma_u & \sigma_u \cdot \sigma_v \\ \sigma_v \cdot \sigma_u & \sigma_v \cdot \sigma_v \end{bmatrix} = \begin{bmatrix} 1 + u^2 & -uv \\ -uv & 1 + v^2 \end{bmatrix}$$

That is, $ds^2 = (1 + u^2)du^2 - 2uvdudv + (1 + v^2)dv^2$.

(b) The length is

$$\int_0^1 \sqrt{(1 + u^2)\dot{u}^2 - 2uv\dot{u}\dot{v} + (1 + v^2)\dot{v}^2} \, dt$$

$$= \int_0^1 \sqrt{(1 + \sinh^2 t) \cosh^2 t - 2 \sinh^2 t \cosh t \cosh^2 t + (1 + \sinh^2 t) \cosh^2 t} \, dt$$

$$= \int_0^1 \sqrt{2 \cosh^2 t} \, dt = \int_0^1 \sqrt{2} \cosh t \, dt = \sqrt{2} \sinh(1)$$