Notes on Series

1. Ratio and Root Tests

Given a series \( \sum_{n=0}^{\infty} a_n \), the **ratio test** or **root test** involve the limits

\[
 r = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} \quad \text{and} \quad r = \lim_{n \to \infty} n^{\sqrt[n]{|a_n|}}. 
\]

In each case, the series converges if \( r < 1 \) and diverges if \( r > 1 \). If \( r = 1 \) or the limit does not exist, the test is inconclusive. In most cases either the ratio test or the root test will work, so you can use whichever one you like better.

2. Power Series

A **power series** centered at \( z = a \) is an infinite series of the form

\[
 \sum_{n=0}^{\infty} c_n (z - a)^n = c_0 + c_1 (z - a) + c_2 (z - a)^2 + c_3 (z - a)^3 + \cdots 
\]

Such a series has a **radius of convergence** \( R \), where \( 0 \leq R \leq \infty \), which can be found using the ratio or root test. Specifically,

\[
 R = \lim_{n \to \infty} \frac{1}{n^{\sqrt[n]{|c_n|}}} = \lim_{n \to \infty} \frac{|c_n|}{|c_{n+1}|}. 
\]

The series converges to a holomorphic function on the disk \( |z - a| < R \) (known as the **disk of convergence**) and diverges for \( |z - a| > R \).

3. Taylor’s theorem

The **Taylor series** for a holomorphic function \( f(z) \) centered at \( z = a \) is the power series

\[
 \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z - a)^n. 
\]

**Taylor’s theorem** states that if \( f(z) \) is holomorphic on the disk \( |z - a| < R \), then \( f(z) \) is equal to the sum of its Taylor series everywhere on this disk.

It follows that the radius of convergence of the Taylor series is the largest radius on which the function \( f(z) \) can be holomorphic. That is, the radius of convergence is the distance from \( a \) to the nearest singularity of \( f(z) \). In the case where \( f(z) \) is a branch of a multivalued function, this should be interpreted as the maximum possible radius around \( a \) on which a branch of the function is holomorphic.
4. Laurent series

A Laurent series centered at \( z = a \) is an infinite series of the form

\[
\sum_{n=\infty}^{\infty} c_n (z - a)^n = \cdots + c_{-2} (z - a)^{1} + c_{-1} (z - a)^{0} + c_0 + c_1 (z - a) + c_2 (z - a)^{2} + \cdots
\]

Such a series has an annulus of convergence \( R_1 < |z - a| < R_2 \). The series converges to a holomorphic function on this annulus, and diverges for \( |z - a| < R_1 \) or \( |z - a| > R_2 \).

Here \( R_2 \) is just the radius of convergence of the power series

\[
c_0 + c_1 (z - a) + c_2 (z - a)^{2} + \cdots
\]

and \( R_1 \) can be found by applying the ratio or root test to the series

\[
c_{-1} (z - a)^{-1} + c_{-2} (z - a)^{-2} + c_{-3} (z - a)^{-3} + \cdots.
\]

Specifically,

\[
R_1 = \lim_{n \to \infty} \sqrt[n]{|c_{-n}|} = \lim_{n \to \infty} \frac{|c_{-n-1}|}{|c_{-n}|} \quad \text{and} \quad R_2 = \lim_{n \to \infty} \frac{1}{\sqrt[n]{|c_{n}|}} = \lim_{n \to \infty} \frac{|c_{n}|}{|c_{n+1}|}
\]

Note that when \( R_1 = 0 \), the function \( f(z) \) is holomorphic on the entire disk \( |z - a| < R_2 \) except at the point \( a \). That is, \( f(z) \) has an isolated singularity at \( z = a \).

Laurent’s theorem states that if a function \( f(z) \) is holomorphic on an annulus \( R_1 < |z-a| < R_2 \), then there exists a Laurent series for \( f(z) \) that converges on this annulus. This includes the case where \( R_1 = 0 \), so a function with an isolated singularity has a Laurent series in a neighborhood of the singularity.

5. Residues

Suppose a holomorphic function \( f(z) \) has an isolated singularity at \( z = a \), and let

\[
f(z) = \sum_{n=-\infty}^{\infty} c_n (z - a)^n
\]

be the Laurent series for \( f(z) \) centered at \( z = a \). Then the residue of \( f(z) \) at \( z = a \) is given by the formula

\[
\text{Res}_{z=a} f(z) = c_{-1}.
\]