An Important Proposition

Most of you proved the following proposition as part of Exercise 1 on Homework 6. But since it wasn’t stated explicitly, and since we’re using it in the most recent set of notes (in the proof of the integral Hölder’s inequality, Theorem 13), I thought I should state and prove it here. You should feel free to use this proposition on Homework 7 and all subsequent homework assignments.

**Proposition**

Let \((X, \mu)\) be a measure space, and let \(f\) be a non-negative measurable function on \(X\). If
\[
\int_X f \, d\mu = 0
\]
then \(f = 0\) almost everywhere.

**PROOF** For each \(n\), let \(E_n = f^{-1}\left((1/n, \infty]\right)\). Note then that each \(E_n\) is measurable. But
\[
0 = \int_X f \, d\mu \geq \int_{E_n} f \, d\mu \geq \int_{E_n} \frac{1}{n} \, d\mu = \frac{\mu(E_n)}{n}
\]
for each \(n\), and therefore each \(\mu(E_n) = 0\). Then
\[
E = \bigcup_{n \in \mathbb{N}} E_n = f^{-1}\left((0, \infty]\right)
\]
is a set of measure zero, and \(f = 0\) on \(E^c\). ■
Corollary

Let $(X,\mu)$ be a measure space, let $f$ be a measurable function on $X$, and let $p \in [1, \infty)$. Then $\|f\|_p = 0$ if and only if $f = 0$ almost everywhere.

**PROOF**  If $f = 0$ almost everywhere, then $|f|^p = 0$ almost everywhere, and it follows that $\|f\|_p = 0$. Conversely, if $\|f\|_p = 0$, then

$$\int_X |f|^p d\mu = 0.$$  

Since $|f|^p \geq 0$, it follows that $|f|^p = 0$ almost everywhere, and hence $f = 0$ almost everywhere.  ■