\( L^p \) Functions

Given a measure space \((X, \mu)\) and a real number \(p \in [1, \infty)\), recall that the \( L^p \)-norm of a measurable function \(f : X \rightarrow \mathbb{R} \) is defined by

\[
\|f\|_p = \left( \int_X |f|^p \, d\mu \right)^{1/p}
\]

Note that the \( L^p \)-norm of a function \(f\) may be either finite or infinite. The \( L^p \) functions are those for which the \( p \)-norm is finite.

**Definition: \( L^p \) Function**

Let \((X, \mu)\) be a measure space, and let \(p \in [1, \infty)\). An \( L^p \) function on \(X\) is a measurable function \(f\) on \(X\) for which

\[
\int_X |f|^p \, d\mu < \infty.
\]

Like any measurable function, and \( L^p \) function is allowed to take values of \( \pm \infty \). However, it follows from the definition of an \( L^p \) function that it must take finite values almost everywhere, so there is no harm in restricting to \( L^p \) functions \(X \rightarrow \mathbb{R}\).

It is easy to see that any scalar multiple of an \( L^p \) is again \( L^p \). Moreover, if \(f\) and \(g\) are \( L^p \) functions, then by Minkowski’s inequality

\[
\|f + g\|_p \leq \|f\|_p + \|g\|_p < \infty
\]

so \(f + g\) is an \( L^p \) function. Thus the set of \( L^p \) functions forms a vector space.

**EXAMPLE 1 \( L^p \) Functions on \([0, 1]\)**

Any bounded function on \([0, 1]\) is automatically \( L^p \) for every value of \(p\). However it is possible for the \( p \)-norm of a measurable function on \([0, 1]\) to be infinite. For example,
let $f : [0, 1] \to \mathbb{R}$ be the function
\[ f(x) = \frac{1}{x} \]
where the value of $f(0)$ is immaterial. Then by the monotone convergence theorem,
\[
\int_{[0,1]} |f| \, dm = \lim_{a \to 0^+} \int_{[a,1]} \frac{1}{x} \, dm(x) = \lim_{a \to 0^+} \left[ \log x \right]_a^1 = \infty
\]
so $f$ is not $L^1$. Indeed, it is easy to check that $f$ is not $L^p$ for any $p \in [1, \infty)$.

A function with a vertical asymptote does not automatically have infinite $p$-norm. For example, if
\[ f(x) = \frac{1}{\sqrt{x}} \]
then $f$ has a vertical asymptote at $x = 0$, but
\[
\int_{[0,1]} |f| \, dm = \lim_{a \to 0^+} \int_{[a,1]} \frac{1}{\sqrt{x}} \, dm(x) = \lim_{a \to 0^+} \left[ 2\sqrt{x} \right]_a^1 = 2.
\]
In general,
\[
\int_{[0,1]} \frac{1}{x^r} \, dm(x) = \begin{cases} 
\infty & \text{if } r \geq 1 \\
\frac{1}{1/(1-r)} & \text{if } r < 1.
\end{cases}
\]
It follows that the function $f(x) = 1/x^r$ is $L^p$ if and only if $pr < 1$, i.e. if and only if $p < 1/r$. For example, $f(x) = 1/\sqrt{x}$ is $L^p$ for all $p \in [1, 2)$, but is not $L^p$ for any $p \in [2, \infty)$.

The last example suggests that it should be harder for a function to be $L^p$ the larger we make $p$. The following proposition confirms this intuition.

**Proposition 1** Relation Between $L^p$ and $L^q$

Let $(X, \mu)$ be a measure space, and let $1 \leq p \leq q < \infty$. If $\mu(X) = 1$, then
\[
\|f\|_p \leq \|f\|_q
\]
for every measurable function $f$. More generally, if $0 < \mu(X) < \infty$, then
\[
\|f\|_p \leq \mu(X)^r \|f\|_q
\]
for every measurable function $f$, where $r = (1/p) - (1/q)$, and hence every $L^q$ function is also $L^p$. 


PROOF The case where $\mu(X) = 1$ is the generalized mean inequality for the $p$-mean and the $q$-mean. For $0 < \mu(X) < \infty$, let $C = \mu(X)$, and let $\nu$ be the measure
\[
d\nu = \frac{1}{C} d\mu.
\]
Then $\nu(X) = 1$, so by the generalized mean inequality
\[
\left( \int_X |f|^p d\mu \right)^{1/p} = C^{1/p} \left( \int_X |f|^p d\nu \right)^{1/p} \leq C^{1/p} \left( \int_X |f|^q d\nu \right)^{1/q} = C^{1/p} C^{-1/q} \left( \int_X |f|^q d\mu \right)^{1/q}.
\]

Note that this proposition only applies in the case where $\mu(X)$ is finite. As the following example shows, the relationship between $L^p$ and $L^q$ functions can be more complicated when $\mu(X) = \infty$.

EXAMPLE 2 Horizontal Asymptotes
Let $f : [1, \infty) \to \mathbb{R}$ be the function
\[
f(x) = \frac{1}{x}.
\]
Then $f$ is not $L^1$, since by the monotone convergence theorem
\[
\int_{[1,\infty)} |f| d\mu = \lim_{b \to \infty} \int_{[1,b]} \frac{1}{x} d\mu(x) = \lim_{b \to \infty} [\log x]_1^b = \infty.
\]
However $f$ is $L^2$, since
\[
\int_{[1,\infty)} |f|^2 d\mu = \lim_{b \to \infty} \int_{[1,b]} \frac{1}{x^2} d\mu(x) = \lim_{b \to \infty} \left[ -\frac{1}{x} \right]_1^b = 1.
\]
In general,
\[
\int_{[1,\infty)} \frac{1}{x^r} d\mu(x) = \begin{cases} 
1/(r-1) & \text{if } r > 1 \\
\infty & \text{if } r \leq 1.
\end{cases}
\]
Thus $f(x) = 1/x^r$ is $L^p$ if and only if $pr > 1$, i.e. if and only if $p > 1/r$.

Thus, for horizontal asymptotes it is easier for a function to be $L^p$ the larger the value of $p$. Intuitively, this is because numbers close to 0 get smaller when taken to a larger power, so $|f|^p$ will be closer to the $x$-axis the larger the value of $p$. 
**$\ell^p$ Sequences**

An important special case of $L^p$ functions is for the measure space $(\mathbb{N}, \mu)$, where $\mu$ is counting measure on $\mathbb{N}$. In this case, a measurable function $f$ on $\mathbb{N}$ is just a *sequence*

$$f(1), \ f(2), \ f(3), \ldots$$

and the Lebesgue integral is the same as the sum of the series

$$\int_{\mathbb{N}} f \, d\mu = \sum_{n \in \mathbb{N}} f(n).$$

The definition of an $L^p$ function on $\mathbb{N}$ takes the following form.

**Definition: $\ell^p$-Norm and $\ell^p$ Sequences**

If $p \in [1, \infty)$, the $\ell^p$-norm of a sequence $\{a_n\}$ of real numbers is defined by the formula

$$\|\{a_n\}\|_p = \left( \sum_{n \in \mathbb{N}} |a_n|^p \right)^{1/p}.$$ 

An $\ell^p$ sequence is a sequence $\{a_n\}$ of real numbers for which

$$\sum_{n \in \mathbb{N}} |a_n|^p < \infty.$$ 

Sequences behave in a similar manner to functions with horizontal asymptotes.

**EXAMPLE 3  P-series**

Recall that the $p$-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if $p > 1$. It follows that the sequence $\{1/n^p\}$ is $\ell^1$ if and only if $p > 1$. For example,

$$\left\{ \frac{1}{n^2} \right\} \text{ is } \ell^1 \quad \text{but} \quad \left\{ \frac{1}{n} \right\} \text{ and } \left\{ \frac{1}{\sqrt{n}} \right\} \text{ are not.}$$

Moreover, since $(1/n^r)^p = 1/n^{rp}$, we find that $\{1/n^r\}$ is $\ell^p$ if and only if $p > 1/r$. Thus

$$\left\{ \frac{1}{n} \right\} \text{ is } \ell^2 \text{ but not } \ell^1,$$
and
\[
\left\{ \frac{1}{\sqrt{n}} \right\} \text{ is } \ell^3 \text{ but not } \ell^2.
\]

All of this is very similar to our analysis of the function \(1/x^p\) on \([1, \infty]\). Indeed, it follows from the integral test that
\[
\int_1^{\infty} \frac{1}{x^p} \, dx < \infty \quad \text{if and only if} \quad \sum_{n=1}^{\infty} \frac{1}{n^p} < \infty
\]
so there is a strong theoretical relationship between these two cases.

**Proposition 2**  Relationship Between \(\ell^p\) and \(\ell^q\)

If \(1 \leq p < q < \infty\), then every \(\ell^p\) sequence is also \(\ell^q\).

**PROOF** Let \(\{a_n\}\) be an \(\ell^p\) sequence. Then
\[
\sum_{n \in \mathbb{N}} |a_n|^p
\]
converges, so it must be the case that \(a_n \to 0\) as \(n \to \infty\). In particular, there exists an \(N \in \mathbb{N}\) such that \(|a_n| < 1\) for all \(n \geq N\). Then \(|a_n|^q < |a_n|^p\) for all \(n \geq N\), so
\[
\sum_{n \in \mathbb{N}} |a_n|^q
\]
converges by the comparison test.

Incidentally, Hölder’s inequality is very interesting for sequences, since it essentially functions as a new convergence test for series.
**Theorem 3  Hölder’s Inequality for Sequences**

Let \( \{a_n\} \) and \( \{b_n\} \) be sequences of real numbers, and let \( p, q \in [1, \infty) \) so that \( 1/p + 1/q = 1 \). If the series

\[
\sum_{n=1}^{\infty} |a_n|^p \quad \text{and} \quad \sum_{n=1}^{\infty} |b_n|^p
\]

both converge, then the series

\[
\sum_{n=1}^{\infty} a_n b_n
\]

converges absolutely, and

\[
\left| \sum_{n=1}^{\infty} a_n b_n \right| \leq \left( \sum_{n=1}^{\infty} |a_n|^p \right)^{1/p} \left( \sum_{n=1}^{\infty} |b_n|^q \right)^{1/q}.
\]

**Corollary 4  Cauchy-Schwarz Inequality for Sequences**

Let \( \{a_n\} \) and \( \{b_n\} \) be sequences of real numbers. If the series

\[
\sum_{n=1}^{\infty} a_n^2 \quad \text{and} \quad \sum_{n=1}^{\infty} b_n^2
\]

both converge, then the series

\[
\sum_{n=1}^{\infty} a_n b_n
\]

converges absolutely, and

\[
\left( \sum_{n=1}^{\infty} a_n b_n \right)^2 \leq \left( \sum_{n=1}^{\infty} a_n^2 \right) \left( \sum_{n=1}^{\infty} b_n^2 \right).
\]
\section*{$L^p$ Completeness}

It is possible to generalize the completeness theorem to $L^p$.

\textbf{Definition: $L^p$ Sequences}

Let $(X, \mu)$ be a measure space, let $\{f_n\}$ be a sequence of measurable functions on $X$, and let $p \in [1, \infty)$.

1. We say that $\{f_n\}$ is an \textbf{$L^p$ Cauchy sequence} if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ so that
   \[ i, j \geq N \implies \|f_i - f_j\|_p < \epsilon. \]

2. We say that $\{f_n\}$ has \textbf{bounded $L^p$-variation} if
   \[ \sum_{n \in \mathbb{N}} \|f_{n+1} - f_n\|_p < \infty. \]

3. We say that $\{f_n\}$ \textbf{converges in $L^p$} to a measurable function $f$ if
   \[ \lim_{n \to \infty} \|f_n - f\|_p = 0. \]

\section*{Theorem 5 $L^p$ Convergence Criterion}

\begin{center}
\textit{Let $(X, \mu)$ be a measure space, and let $\{f_n\}$ be a sequence of measurable functions on $X$ with bounded $L^p$-variation. Then $\{f_n\}$ converges pointwise almost everywhere to a measurable function $f$, and $f_n \to f$ in $L^p$.}
\end{center}

\textbf{PROOF} Let
\[ M = \sum_{n \in \mathbb{N}} \|f_{n+1} - f_n\|_p < \infty. \]
and let
\[ g = \sum_{n=1}^{\infty} |f_{n+1} - f_n| \quad \text{and} \quad g_N = \sum_{n=1}^{N} |f_{n+1} - f_n| \]
for each $N \in \mathbb{N}$. By Minkowski’s inequality,
\[ \|g_N\|_p \leq \sum_{n=1}^{N} \|f_{n+1} - f_n\|_p \leq M \]
for all $N \in \mathbb{N}$. By the monotone convergence theorem, it follows that

$$\int_X g^p \, d\mu = \int_X \lim_{N \to \infty} g^p_N \, d\mu = \lim_{N \to \infty} \int_X g^p_N \, d\mu = \lim_{N \to \infty} \|g_N\|^p_p \leq M^p < \infty.$$ 

From this we conclude that $g(x) < \infty$ for almost all $x \in X$, so $\{f_n(x)\}$ has bounded variation for almost all $x \in X$, and hence $\{f_n(x)\}$ converges pointwise almost everywhere.

Let $f$ be the pointwise limit of the sequence $\{f_n\}$, and note that for each $n \in \mathbb{N}$,

$$f - f_n = \lim_{N \to \infty} f_{N+1} - f_n = \lim_{N \to \infty} \sum_{k=n}^{N} (f_{k+1} - f_k) = \sum_{k=n}^{\infty} (f_{k+1} - f_k)$$

almost everywhere. Then

$$|f - f_n|^p = \left| \sum_{k=n}^{\infty} (f_{k+1} - f_k) \right|^p \leq \left( \sum_{k=n}^{\infty} |f_{k+1} - f_k| \right)^p \leq g^p$$

almost everywhere, so by the dominated convergence theorem

$$\lim_{n \to \infty} \int_X |f - f_n|^p \, d\mu = \int_X \lim_{n \to \infty} |f - f_n|^p \, d\mu = 0.$$

Thus $f_n \to f$ in $L^p$. $lacksquare$

$L^p$ completeness follows easily. We leave the proof to the reader.

**Theorem 6**  \textit{L}$^p$ Completeness

\begin{center}
\begin{tabular}{|l|}
\hline
\textit{Let $(X, \mu)$ be a measure space, and let $\{f_n\}$ be an $L^p$ Cauchy sequence on $X$. Then $\{f_n\}$ converges in $L^p$ to some measurable function $f$ on $X$.} \\
\hline
\end{tabular}
\end{center}
The $L^\infty$ Norm

It is possible to extend the $L^p$ norms in a natural way to the case $p = \infty$.

**Definition: $L^\infty$-Norm**

Let $(X, \mu)$ be a measure space, and let $f$ be a measurable function on $X$. The $L^\infty$-norm of $f$ is defined as follows:

$$\|f\|_\infty = \min \{M \in [0, \infty] \mid |f| \leq M \text{ almost everywhere} \}.$$  

We say that $f$ is an $L^\infty$ function if $\|f\|_\infty < \infty$.

Note that the set

$$\{M \in [0, \infty] \mid |f| \leq M \text{ almost everywhere}\}$$

really does have a minimum element, for if $|f| \leq M + 1/n$ almost everywhere for all $n \in \mathbb{N}$, then it follows that $|f| \leq M$ almost everywhere.

The $L^\infty$-norm $\|f\|_\infty$ is sometimes called the essential supremum of $|f|$, and $L^\infty$ functions are sometimes said to be essentially bounded or bounded almost everywhere. Note that a continuous function on $\mathbb{R}$ is $L^\infty$ if and only if it is bounded, in which case $\|f\|_\infty$ is equal to the supremum of $|f|$.

Much of what we have done for $p \in [1, \infty)$ also works for $p = \infty$. We list some of the results, and leave the proofs to the reader:

**Minkowski’s Inequality.** If $f$ and $g$ are $L^\infty$ functions, then $f + g$ is $L^\infty$, and

$$\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty.$$  

**Hölder’s Inequality.** If $f$ is an $L^1$ function and $g$ is an $L^\infty$ function, then $fg$ is Lebesgue integrable and

$$|\langle f, g \rangle| \leq \|f\|_1 \|g\|_\infty.$$  

**$L^\infty$ Convergence.** If $\{f_n\}$ is a sequence of functions, we say that $\{f_n\}$ converges in $L^\infty$ to a function $f$ if

$$\lim_{n \to \infty} \|f_n - f\|_\infty = 0.$$  

This turns out to be the same as uniform convergence almost everywhere, i.e. $f_n \to f$ in $L^\infty$ if and only if there exists a set $Z$ of measure zero such that $f_n \to f$ uniformly on $Z^c$. 
L∞ Completeness. If \( \{ f_n \} \) is an \( L^\infty \) Cauchy sequence of measurable functions, then \( \{ f_n \}_\infty \) converges in \( L^\infty \) to some measurable function \( f \).

Relation Between \( L^\infty \) and \( L^p \) If \( \mu(X) = 1 \), then \( \| f \|_p \leq \| f \|_\infty \) for any measurable function \( f \) on \( X \). More generally, if \( 0 < \mu(X) < \infty \) then
\[
\| f \|_p \leq \mu(X)^{1/p} \| f \|_\infty
\]
for all \( p \), so any \( L^\infty \) function on \( X \) is also \( L^p \) for all \( p \in [1, \infty) \).

In the case of sequences, the \( L^\infty \) norm takes the following form.

**Definition: \( \ell^\infty \)-Norm**

Let \( \{ a_n \} \) be a sequence of real numbers. The \( \ell^\infty \)-norm of \( \{ a_n \} \) is defined as follows:
\[
\| \{ a_n \} \|_\infty = \sup_{n \in \mathbb{N}} |a_n|
\]

Thus an \( \ell^\infty \) sequence is the same as a bounded sequence. Note that if \( p \in [1, \infty) \), then any \( \ell^p \) sequence must be \( \ell^\infty \), since any \( \ell^p \) sequence must converge to zero.

**Exercises**

For the following exercises, let \((X, \mu)\) be a measure space.

1. Let \( f : [0, \infty) \to \mathbb{R} \) be the function \( f(x) = e^{-x} \). For what values of \( p \) is \( f \) an \( L^p \) function?

2. Let \( f : (0, \infty) \to \mathbb{R} \) be the function
\[
f(x) = \begin{cases} 
x^{-1/3} & 0 < x < 1, \\
x^{-1/2} & 1 \leq x < \infty.
\end{cases}
\]
For what values of \( p \) is \( f \) an \( L^p \) function?

3. Let \( f : [0, 1] \to [0, \infty] \) be the function \( f(x) = -\log x \), with \( f(0) = \infty \).
   
   (a) Show that \( f \) is \( L^1 \).

   (b) Show that \( f \) is \( L^p \) for all \( p \in [1, \infty) \). (Hint: Substitute \( u = 1/x \).)
4. For what values of $p$ is
\[ \left\{ \frac{1}{(n^2 + 1)^{1/3}} \right\} \]
an $\ell^p$ sequence?

5. For what values of $p$ is
\[ \left\{ \frac{1}{\sqrt{n} \log n} \right\} \]
an $\ell^p$ sequence?

6. Prove that every $L^p$ Cauchy sequence has a subsequence of bounded $L^p$-variation.

7. Prove the $L^p$ completeness theorem (Theorem [6]).

8. If $f$ and $g$ are measurable functions on $X$, prove that $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

9. If $f$ is an $L^1$ function on $X$ and $g$ is an $L^\infty$ function on $X$, prove that $fg$ is Lebesgue integrable and $|\langle f, g \rangle| \leq \|f\|_1 \|g\|_\infty$.

10. Let $\{f_n\}$ be a sequence of measurable functions on $X$, and let $f$ be a measurable function on $X$. Prove that $f_n \to f$ in $L^\infty$ if and only if $f_n \to f$ uniformly almost everywhere.

11. If $0 < \mu(X) < \infty$ and $f$ is a measurable function on $X$, prove that
\[ \|f\|_p < \mu(X)^{1/p} \|f\|_\infty \]
for all $p \in [1, \infty)$.

12. Prove that every $L^\infty$ Cauchy sequence of measurable functions converges uniformly almost everywhere.