Thompson-Like Groups Acting on Julia Sets

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In the 1960’s, Richard J. Thompson defined three infinite groups:

- $F$ acts on the interval.
- $T$ acts on the circle.
- $V$ acts on the Cantor set.
Definition of $F$

A **dyadic subdivision** of $[0, 1]$ is any subdivision obtained by repeatedly cutting intervals in half:
Definition of $F$

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![Diagram](image-url)
A *dyadic subdivision* of $[0, 1]$ is any subdivision obtained by repeatedly cutting intervals in half:

![Division of the interval [0, 1] into dyadic subdivisions](image-url)
Definition of $F$

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Definition of $F$

A **dyadic subdivision** of $[0, 1]$ is any subdivision obtained by repeatedly cutting intervals in half:

![Diagram of dyadic subdivision]

The partition points are always dyadic fractions.
A **dyadic rearrangement** of \([0, 1]\) is a PL homeomorphism that maps linearly between the intervals of two dyadic subdivisions:

The set of all dyadic rearrangements of \([0, 1]\) is **Thompson’s group \(F\)**.
Definition of $F$

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The set of all dyadic rearrangements is *Thompson’s group $F$*. 
Definitions of $T$ and $V$

**Thompson’s Group $T$** acts on a circle.

**Thompson’s Group $V$** acts on a Cantor set.
Properties of the Thompson Groups

- $T$ and $V$ are **infinite, finitely presented simple groups**.

- $F$ is finitely presented but not simple.

- Finiteness properties: All three have **type $F_\infty$**. (Brown & Geoghegan, 1984)

- Geometry: All three act properly and isometrically on CAT(0) cubical complexes. (Farley, 2003)
Generalizations

Basic Question:
Why are there *three* Thompson groups?
Generalizations

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Generalizations:
- $F(n)$, $T(n)$, and $V(n)$ (Higman 1974, Brown 1987)
- Other PL groups (Bieri & Strebel 1985, Stein 1992)
- Diagram Groups (Guba & Sapir 1997)
- “Braided” $V$ (Brin 2004, Dehornoy 2006)
- $2V$, $3V$, $\ldots$ (Brin 2004)
Self-Similarity
Self-Similarity

$F$ depends on the self-similarity of the interval:

Interval

0 1
Self-Similarity

$F$ depends on the **self-similarity** of the interval:

![Graph showing self-similarity of an interval](image)
Self-Similarity

$F$ depends on the **self-similarity** of the interval:

Some fractals have this same self-similar structure:

Koch Curve
Self-Similarity

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Some fractals have this same self-similar structure:
Self-Similarity

Thompson’s group $F$ acts on such a fractal by piecewise similarities.
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**Idea:** Find Thompson-like groups associated to other self-similar structures.
Self-Similarity

Thompson’s group $F$ acts on such a fractal by piecewise similarities.

**Idea:** Find Thompson-like groups associated to other self-similar structures.

But where can we find other self-similar structures?
Julia Sets

Every rational function on the Riemann sphere has an associated *Julia set.*
Example: The Julia set for \( f(z) = z^2 - 1 \) is called the *Basilica*.

It is the simplest example of a fractal Julia set.
Julia Sets: The Basilica

The Basilica has a “self-similar” structure.
Invariance of the Basilica under $z^2 - 1$

The Basilica maps to itself under $f(z) = z^2 - 1$. 
Julia Sets: The Basilica

The Basilica has a *conformally self-similar* structure.
The Basilica Group
The Plan

Let’s try to construct a Thompson-like group that acts on the Basilica.
The Plan

Let’s try to construct a Thompson-like group that acts on the Basilica.

<table>
<thead>
<tr>
<th>Interval</th>
<th>Basilica</th>
</tr>
</thead>
<tbody>
<tr>
<td>linear map</td>
<td>conformal map</td>
</tr>
<tr>
<td>dyadic subdivision</td>
<td>???</td>
</tr>
</tbody>
</table>
Structure of the Basilica

**Terminology**: Each of the highlighted sets below is a *bulb*. 
Structure of the Basilica

The Basilica is the union of two bulbs.
Structure of the Basilica

Each bulb has three parts.
Structure of the Basilica

Each bulb has three parts.
Structure of the Basilica

Each edge also has three parts.
Structure of the Basilica

Each edge also has three parts.
Allowed Subdivisions of the Basilica

**Allowed subdivision:**
Start with the base and repeatedly apply the two subdivision moves.

Base

![Diagram of Allowed Subdivisions](image-url)
Rearrangements of the Basilica

A **rearrangement** is a homeomorphism that maps conformally between the pieces of two allowed subdivisions.

**Example 1**

**Domain:**

**Range:**
Example 1
Example 2

Domain: C → B → A

Range: E → D → B → F → E

A → E → D → B → A
Example 2
The Group $T_B$

Let $T_B$ be the group of all rearrangements of the Basilica.

Theorem

1. $T_B$ contains isomorphic copies of Thompson’s group $T$. 
The Group $T_B$

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**Theorem**

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2. $T$ contains an isomorphic copy of $T_B$. 
The Group $T_B$

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1. $T_B$ contains isomorphic copies of Thompson’s group $T$.
2. $T$ contains an isomorphic copy of $T_B$.
3. $T_B$ is generated by four elements.
The Group $T_B$

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Theorem

1. $T_B$ contains isomorphic copies of Thompson’s group $T$.
2. $T$ contains an isomorphic copy of $T_B$.
3. $T_B$ is generated by four elements.
4. $T_B$ has a simple subgroup of index two.
Aside: Graphs and Diagram Groups

Everything here is combinatorial.

An allowed subdivision can be represented by a directed graph:
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An allowed subdivision can be represented by a directed graph:
Aside: Graphs and Diagram Groups

There are two replacement rules for these graphs:

These constitute a graph rewriting system.
Aside: Graphs and Diagram Groups

All we really need to define $T_B$ are the graph rewriting system:

$$\text{Rule 1} \quad \rightarrow \quad \text{Rule 2}$$

and the base graph:

Base Graph
Victor Guba and Mark Sapir defined *diagram groups*:

- Generalization of Thompson’s groups
- Uses string rewriting systems.

$T_B$ is similar to a diagram group, except that it uses graph rewriting.

**Theorem (Farley).** *Every diagram group over a finite string rewriting system acts properly by isometries on a CAT(0) cubical complex.*

A similar construction gives a natural CAT(0) cubical complex on which $T_B$ acts properly by isometries.
Other Julia Sets
Julia Sets

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The Mandelbrot Set

Julia sets for quadratic polynomials $f(z) = z^2 + c$ are parameterized by the Mandelbrot set:
The Mandelbrot Set

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The Mandelbrot Set

Points in the interior of the Mandelbrot set are called *hyperbolic*.
The Mandelbrot Set

Hyperbolic points from the same interior region give Julia sets with the same structure.
The Mandelbrot Set

We can construct a Thompson-like group $T_J$ for each of these regions. (Hubbard tree $\rightarrow$ Graph rewriting system)
The Mandelbrot Set

We can construct a Thompson-like group $T_J$ for each of these regions. (Hubbard tree $\rightarrow$ Graph rewriting system)
The Airplane Group

Let $T_A$ be the group of rearrangements of the airplane Julia set.

**Theorem.**

1. $T_A$ has a simple subgroup of index 3.
2. $T_A$ has type $F_\infty$.

The proof of (2) involves discrete Morse Theory on the CAT(0) cubical complex for $T_A$. 
Questions

- Are all the $T_J$ finitely generated? Is there a uniform method to find a generating set?
- Which $T_J$ are finitely presented? Which have type $F_\infty$?
- Which of these groups are virtually simple?
- What is the relation between these groups? For which Julia sets $J$ and $J'$ does $T_J$ contain an isomorphic copy of $T_{J'}$?
- For which rational Julia sets can we construct a Thompson-like group? Are there Thompson-like groups associated to other families of fractals?
The End