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Critical Points and the Angle Defect

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Abstract. In a 1967 paper, Banchoff described a theory of critical points and curvature for polyhedra embedded in Euclidean space. For each convex cell complex K in $\mathbb{R}^n$, and for each linear map $h : \mathbb{R}^n \to \mathbb{R}$ satisfying a simple generality criterion, he defined an index for each vertex of K with respect to the map h, and showed that these indices satisfy two properties: (1) for each map h, the sum of the indices at all the vertices of K equals $\chi(K)$; and (2) for each vertex of K, the integral of the indices of the vertex with respect to all such linear maps equals the standard polyhedral notion of curvature of K at the vertex. In a previous paper, the author defined a different approach to curvature for arbitrary simplicial complexes, based upon a more direct generalization of the angle defect. In the present paper we present an analog of Banchoff's theory that works with our generalized angle defect.

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1. Introduction

In [1–3], Banchoff described a very nice theory of critical points and curvature for polyhedra embedded in Euclidean space. The 'Morse functions' Banchoff used are linear maps to lower dimensional Euclidean spaces (in particular, to one-dimensional Euclidean spaces in [1], which is the case in which we are interested). Banchoff's approach is a polyhedral version of the approach to critical points and curvature due to Kuiper in [4].

In [1], Banchoff took a convex cell complex K in $\mathbb{R}^n$, and for each linear map $h: \mathbb{R}^n \to \mathbb{R}$ satisfying a simple generality criterion, he defined an index for each vertex of K with respect to the map h, and showed that these indices satisfy two main properties: (1) the sum of the indices at all the vertices of K, with respect to a given map h, equals $\chi(K)$; and (2) for each vertex of K, the integral of the indices of the vertex with respect to all such linear maps equals the curvature of K at the vertex.

The type of polyhedral curvature used by Banchoff in [1] is a well known definition of curvature of embedded polyhedra that generalizes the classical angle defect. For a polyhedral surface M^2 and a vertex v of M, the angle defect at v is $d_v = 2\pi - \sum \alpha_i$, where the α_i are the angles of the triangles containing v. This curvature function goes back at least as far as Descartes (see [5]), and it satisfies all the standard properties one would expect a curvature function on polyhedra to satisfy, including a polyhedral Gauss–Bonnet Theorem, which says $\sum_v d_v = 2\pi\chi(M^2)$, where the summation is over all the vertices of M^2. The type of curvature used in [1] reduces to the angle defect for a polyhedral surface, and it satisfies appropriately nice properties, including a Gauss–Bonnet Theorem, in all dimensions. We will refer to this type of curvature as 'standard curvature'. This curvature has been studied widely, for example, in [6–10] . This approach to generalizing the angle defect, which is based on exterior angles, is simple to define, and it's convergence properties has been well studied. In standard curvature, all the curvature is concentrated at the vertices, as in the case for the classical angle defect of polyhedral surfaces.

It turns out that standard curvature is not the only possible generalization of the classical angle defect to arbitrary polyhedra in higher dimensions. In [11] we defined a different approach to curvature for arbitrary simplicial complexes, which we call stratified curvature. Our approach is based on the angle defect idea, but extended to non-manifolds via a simple topological decomposition of each simplicial complex. The angle defect (also known as the angle deficiency), has been studied in the case of convex polytopes by a number of combinatorialists, for example [12, 13]; more generally, for the wider study of angle sums in convex polytopes and beyond, see for example [14, Chapter 14, 15–18]. In [19] a Gauss–Bonnet type theorem (also referred to as Descartes' Theorem) is proved for the angle defect in polytopes with underlying spaces that are manifolds. The angle defect for convex polytopes resembles the classical angle defect for polyhedral surfaces much more closely than does standard curvature. In contrast to standard curvature, which is concentrated at the vertices, the angle defect for convex polytopes is found at each simplex of co-dimension at least 2 (it can be defined for all simplices, but the angle defect at a co-dimension 0 or 1 simplex will always be zero). The angle defect for convex polytopes satisfy various nice properties, such as Gauss–Bonnet type theorem. One treatment of curvature of polyhedra that has some of the advantages of all the approaches cited above is in [3], which uses curvatures functions based on critical points (similarly to [1]), but this time using projection maps $\mathbb{R}^n \to \mathbb{R}^m$, which leads to curvature functions related to the Grassman angles of [13], and which are located at all simplices, and which directly generalizes standard curvature; moreover, an angle defect type formula for curvature is obtained using projection maps $\mathbb{R}^n \to \mathbb{R}^{n-1}$.

In [11, Section 4], we take the approach to curvature for arbitrary simplicial complexes that is most directly comparable to the combinatorial authors listed above. In [11, Section 4] we referred to this approach by the unfortunate name of 'modified stratified curvature', which really misses the point that in this approach we are really still working with a pure angle defect. Hence, in the present paper, we will use the better name of 'generalized angle defect' (which is also used in [20]). In [11, Section 3] we defined a curvature function called 'stratified curvature', which concentrated all the angle defects at the vertices of simplicial complexes; doing so was not very natural, and will not be used in the present paper, though it was useful in comparing our approach to standard curvature.

A detailed comparison of standard curvature with both stratified curvature and the generalized angle defect (which are just variants of each other) may be found in [11, Section 4]. We mention here, however, that all these approaches satisfy some of the basic properties that one would expect of curvature, such as being locally defined, invariant under local isometries, and satisfying a Gauss–Bonnet type theorem (though the Gauss–Bonnet Theorem for stratified curvature and the generalized angle defect uses a modified Euler characteristic rather than the standard Euler characteristic, as discussed in [11, Section 2]). One property where the different types of curvature do not behave similarly is that standard curvature is identically zero for any odd-dimensional polyhedral manifold (as stated without proof in [1, Section 5]), but the analogous property does not hold for stratified curvature and the generalized angle defect (as discussed in [20], where a modified version of generalized angle defect is shown to satisfy this property).

The purpose of the present paper is to show that an analog of Banchoff's theory of critical points for embedded polyhedra, as found in [1–2], can be obtained for the generalized angle defect. For the sake of completeness, we mention here some of the similarities and differences of our approach to that found in these two papers of Banchoff, as well as his later [3], and the recent combinatorial Morse theory of Forman, found in [21] and many other papers.

First, we note that in all three papers of Banchoff that have been cited, the general setting is convex cell complexes, and in the work of Forman the most general setting is CW complexes, whereas our approach is restricted to simplicial complexes. As in Banchoff's work, our simplicial complexes are all embedded in Euclidean space, as opposed to Forman's approach, in which abstract simplicial complexes are used. As in [1, 2], the type of 'Morse functions' that we use will be projection maps from $\mathbb{R}^m$ onto one-dimensional linear subspaces; each such projection map corresponds to a vector in S^{m-1}. We cannot use all such projection maps, because of some degenerate cases, and so we need to rule out some 'bad' unit vectors; in [1] some unit vectors are also ruled out, though our criteria for disallowed unit vectors is different from [1]. In both our treatment and in [1], the set of disallowed unit vectors has measure zero in S^{m-1}, and therefore can be safely ignored for our purposes. In [3] projection maps from $\mathbb{R}^m$ to linear subspaces off all dimensions are used; we do not treat such maps. In [21], where simplicial complexes are not assumed to be embedded, the 'Morse functions' are not projections of $\mathbb{R}^m$ onto linear subspaces, but are rather purely combinatorial functions, and as such are rather different from the approach we take.

Suppose we are given a simplicial complex K in $\mathbb{R}^m$, and a unit vector $\xi \in S^{m-1}$. We will define the index of each simplex of co-dimension at least 2 of K with respect to the projection map h_ξ from $\mathbb{R}^m$ onto the one-dimensional subspace generated by ξ. In [1, 2], the index of a vertex of a simplicial complex with respect to ξ, denoted $a(v, \xi)$ in [1] and $i(v, \xi)$ in [2], is defined in terms of the relative values under h_ξ of the vertices of the simplices containing v (the definition is formulated slightly differently in the two papers, though the two approaches are equivalent). Banchoff's simple and elegant approach works very nicely with respect to standard curvature, because that curvature is defined in terms of exterior angles, and Banchoff's definition of the index of a vertex is naturally related to exterior angles. Our approach to curvature uses interior angles, and, as a result, we cannot use Banchoff's simple definition of the index, though we also use a definition that is expressed in terms of the values under h_ξ of certain vertices. The definition of our index, which will be given in Equation (11) below, more closely resembles the definition Banchoff gives for his index in [2, p. 478], which is for simplicial surfaces only, than it resembles the definition he gives for his index in of [1, p. 246], which is for all simplicial complexes. Our approach can be seen as an alternative generalization of the formulation in [2] to arbitrary simplicial complexes. Also, we note that in [1] the index is defined only at the vertices of a simplicial complex (which makes sense because standard curvature is defined only at the vertices), whereas we define an index at every simplex of co-dimension at least 2 (and we could define the index of simplices of codimenion 1 or 0 to be zero); in [3] the index is also defined for all simplices, not just vertices.

To make this paper self-contained, we start, in Section 2, with a brief review all needed definitions and theorems from [11], leaving all the details to that paper. We give all new definitions and statements of results in Section 3, and then give proofs in Section 4.

2. Review of the Generalized Angle Defect

We give here a very brief summary of those definitions and statements of results from [11] that we need; we refer the reader to the original paper for proofs and further discussion. Throughout this paper, we will assume that all simplicial complexes are finite, of dimension at least 2, and are in Euclidean space. (Whereas in [11] we allow for a certain class of nonembedded simplicial complexes, here for convenience we look only at actual simplicial complexes in Euclidean space.)

For the duration of this section, let K be an n-dimensional simplicial complex in Euclidean space. If η and σ are simplices in K, we write $\eta \prec \sigma$ to indicate that η is a face of σ. As usual, we let $|K|$ denote the underlying space of K.

For the sake of convenience, we adopt the convention that we normalize all angles so that the volume of the unit $(n-1)$-sphere in $(n-1)$-measure is 1 in all dimensions. For any n-simplex σ^n in Euclidean space, and any i-face η^i of σ^n, let $\alpha(\eta^i, \sigma^n)$ denote the solid angle in σ^n along η^i, where by normalization such an angle is always a number in [0, 1].

DEFINITION 2.1. For each nonnegative integer i, let T_i denote the open cone on i points; alternatively, T_i is the space obtained by gluing together i copies of the half open interval $[0, 1)$ at the point $\{0\}$ in each. We take T_0 to be a single point. See Figure 1. Let $P_{n,i}$ denote the space $P_{n,i} = T_i \times \mathbb{R}^{n-1}$. See Figure 2. If $*$ denotes the cone point of T_i, we call $\{*\} \times \mathbb{R}^{n-1} \subseteq P_{n,i}$ the *apex set* of T_i.

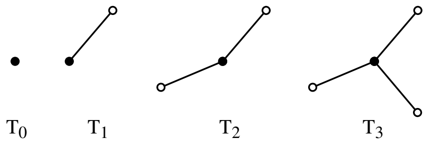

Figure 1.

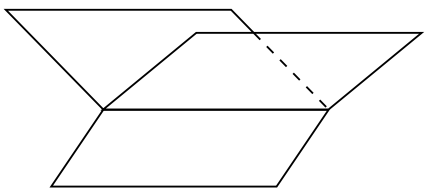

Figure 2.

Observe that $P_{n,i}$ is not homeomorphic to $P_{n,j}$ when $i \neq j$. For our next definition, and from now on, we will need to think of simplices as open (and hence disjoint).

DEFINITION 2.2. Let K be an n-dimensional simplicial complex. For each non-negative integer r such that $r \neq 2$, we define the subset $C_r^n(K)$ of $|K|$ by $C_r^n(K) = \{x \in |K| \mid x$ has a neighborhood homeomorphic to $P_{n,r}$, where the homeomorphism takes x to the apex set of $P_{n,r}\}$.

Define

$$C_2^n(K) = |K| - \bigcup_{r \neq 2} C_r^n(K).$$

EXAMPLE 2.3. Consider the two-dimensional simplicial complex K shown in Figure 3. The set $C_2^n(K)$ consists of the interiors of the three triangles together with the vertex w. The set $C_1^n(K)$ is the union of the boundaries of the triangles with w removed. Note that $C_r^n(K) = \emptyset$ for $r \neq 1, 2$.

Remark 2.4. (1) The sets $C_r^n(K)$ are well defined, because each $x \in |K|$ can have a neighborhood homeomorphic to $P_{n,r}$ (where the homeomorphism takes x to the apex set of $P_{n,r}$) for at most one number $r \neq 2$. Moreover, the sets $C_r^n(K)$ are well defined up to homeomorphism of $|K|$.

(2) Because K is a finite simplicial complex, there is some positive integer P such that $C_r^n(K) = \emptyset$ for all $r > P$.

(3) The sets $C_r^n(K)$ are disjoint, and cover $|K|$. For each $r \neq 2$, the set $C_r^n(K)$ is an $(n-1)$-manifold without boundary. Moreover, each set $C_r^n(K)$ is the union of (open) simplices of K, since all points in any simplex of K have homeomorphic neighborhoods in $|K|$ (if the neighborhoods are taken small enough). If $\sigma \in K$, then $\sigma \subseteq C_r^n(K)$ for some unique integer r.

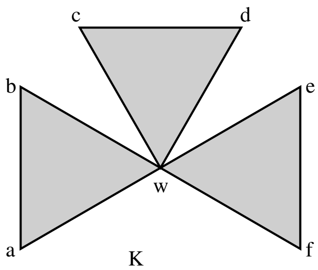

Figure 3.

DEFINITION 2.5. Let K be an n-dimensional simplicial complex. For each simplex $\sigma \in K$, we define the number $T_n(\sigma)$ by $T_n(\sigma) = r/2$, where $\sigma \in C_r^n(K)$ for some unique integer r.

The following definition was originally given in [11, Section 4], though here we use the better name given given below (and also used in [20], as discussed in Section 1). In contrast to standard curvature, which in all dimensions is concentrated at the vertices (see, for example, [1, 8]), our approach has curvature at all simplices (though the nonzero curvature is always at simplices of co-dimension at least 2), similarly to the combinatorial approach (see for example [12, 13]), as well as the geometric approach of [3].

DEFINITION 2.6. Let K be an n-dimensional simplicial complex, and let η^i be an i-simplex of K, where $0 \leqslant i \leqslant n-2$. The *generalized angle defect* at η^i is the number $D_n(\eta^i)$ defined by $D_n(\eta^i) = T_n(\eta^i) - \sum_{\sigma^n \succ \eta^i} \alpha(\eta^i, \sigma^n)$, where the summation is over all n-simplices σ^n which have η^i as a face.

EXAMPLE 2.7. We continue Example 2.3. Assume that all three triangles in K are equilateral. By normalization of angles, each angle in an equilateral triangle is $1/6$. Then

$$D_2(w) = T_2(w) - \sum_{\sigma^2 \succ w} \alpha(w, \sigma^2) = 1 - 3 \cdot \frac{1}{6} = \frac{1}{2},$$

and

$$D_2(a) = T_2(a) - \sum_{\sigma^2 \succ a} \alpha(a, \sigma^2) = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}.$$

Clearly the generalized angle defect at each of b, c, d, e, f is the same as at a.

In the Gauss–Bonnet type theorem proved in [11], rather than using the standard Euler characteristic, we used the following variant of the Euler characteristic. We will use this new characteristic in the present paper as well.

DEFINITION 2.8. Let K be an n-dimensional simplicial complex in $\mathbb{R}^m$. The number $\chi^s(K)$ is defined by

$$\chi^s(K) = \sum_{\eta \in K} T_n(\eta)(-1)^{\dim \eta}.$$

The above definition is a particular case of the weighted Euler characteristics discussed in [22, 23]. In the notation of those two papers, the symbol $\chi^s(K)$ would be written $\chi(K, T)$, but we will not need this latter notation, because we will never use a different weight on the simplices.

EXAMPLE 2.9. We continue Example 2.3. Clearly $\chi(K) = 1$, but

$$\chi^s(K) = 3 \cdot 1 - 9 \cdot \frac{1}{2} + 6 \cdot \frac{1}{2} + 1 \cdot 1 = \frac{5}{2}.$$

Observe that the sum of the generalized angle defects at the vertices of K, as computed in Example 2.7, equals 5/2, as expected by the Gauss–Bonnet theorem proved in [11].

As discussed in [11, Section 2], it can be seen that $\chi^s(K)$ is a homeomorphism invariant of $|K|$, but it is not a homotopy type invariant.

3. Stratified Morse Index

We will use the following notation. If η^i is a simplex in $\mathbb{R}^m$, we let $V(\eta^i)$ denote the i-dimensional vector subspace of $\mathbb{R}^m$ that is parallel to the i-plane spanned by η^i. We can think of $V(\eta^i)$ as a copy of $\mathbb{R}^i$, we can think of S^{i-1} as the set of unit vectors in $V(\eta^i)$, and we can think of η^i as sitting in $V(\eta^i)$ by translation.

If T is a vector subspace of $\mathbb{R}^m$, we let $h_T \colon \mathbb{R}^m \to T$ denote orthogonal projection onto T. Let ξ be a vector in $\mathbb{R}^m$. For convenience we will write h_ξ instead of $h_{V(\xi)}$. As in [1], we will think of $V(\xi)$ as a copy of the real number line, and can therefore think of $h_\xi(x)$ for each $x \in \mathbb{R}^m$ as a real number, rather than a vector. If ξ is a unit vector, then clearly $h_\xi(x) = x \cdot \xi$ for all $x \in \mathbb{R}^m$.

If K is an n-dimensional simplicial complex in $\mathbb{R}^m$, then the type of "Morse functions" on K that we use will be projection maps of the form $h_\xi : \mathbb{R}^m \to V(\xi)$, for almost all unit vectors $\xi \in S^{m-1}$. We cannot use the projection map h_ξ for all unit vectors ξ, because of some degenerate cases, and so we need to rule out some "bad" unit vectors, as done in the following definition. The set of disallowed unit vectors has measure zero in the unit sphere.

DEFINITION 3.1. Let K be an n-dimensional simplicial complex in $\mathbb{R}^m$, and let $\xi \in S^{m-1}$. We say that ξ is an *allowable vector* with respect to K if the following criteria hold. Let σ^n be any n-simplex of K. For convenience let $T = V(\sigma^n)$. Then we require that $h_T(\xi)$ is not the zero vector, and that $h_T(\xi)$ is not contained in $V(\eta^{n-1})$ for any $(n-1)$-face η^{n-1} of σ^n.

LEMMA 3.2. *Let K be an n-dimensional simplicial complex in $\mathbb{R}^m$. Then the set of allowable vectors in S^{m-1} with respect to K is an open dense subset of S^{m-1}, and the set of non-allowable vectors in S^{m-1} with respect to K has measure zero.*

Proof. Let σ^n be an n-simplex of K and let η^{n-1} be an $(n-1)$-face of σ^n. Let $U(\sigma^n, \eta^{n-1}) = V(\sigma^n)^{\perp} \oplus V(\eta^{n-1})$. It is simple to see that $U(\sigma^n, \eta^{n-1})$ is an $(m-1)$-dimensional vector subspace of $\mathbb{R}^m$, and hence $U(\sigma^n, \eta^{n-1}) \cap S^{m-1}$ is a closed subset of measure zero of S^{m-1}. The set of all non-allowable vectors in S^{m-1} with respect to K is precisely the union of all the sets $U(\sigma^n, \eta^{n-1}) \cap S^{m-1}$. The result follows immediately. □

The following definition makes sense because of the definition of allowable vectors.

DEFINITION 3.3. Let K be an n-dimensional simplicial complex in $\mathbb{R}^m$, let σ^n be an n-simplex of K, and let $\xi \in S^{m-1}$ be an allowable vector with respect to K. For convenience let $T = V(\sigma^n)$. We then define ξ_{σ^n} to be the unit vector in T defined by

$$\xi_{\sigma^n} = \frac{h_T(\xi)}{\|h_T(\xi)\|}.$$

Remark 3.4. Let K be an n-dimensional simplicial complex in $\mathbb{R}^m$, let σ^n be an n-simplex of K, and let $\xi \in S^{m-1}$ be an allowable vector with respect to K. We observe that ξ_{σ^n} is an allowable vector in S^{n-1} with respect to σ^n (thought of as sitting in $V(\sigma^n)$).

We now want to define the index of each simplex of co-dimension at least 2 of a simplicial complex, with respect to a projection map of the form h_ξ, where ξ is an allowable vector. Analogously to [1, 2], the index is expressed in terms of the values under h_ξ of certain vertices. Our index is given in Equation (11), though we start with some preliminaries.

DEFINITION 3.5. Let K be an n-dimensional simplicial complex in $\mathbb{R}^m$, let $\sigma^n = \langle a_0, \ldots, a_{n-1}, b\rangle$ be an n-simplex of K, let $\tau^{n-1} = \langle a_0, \ldots, a_{n-1}\rangle$, and let $\xi \in S^{m-1}$ be an allowable vector with respect to K. We use the abbreviations $x_i = a_i - a_0$ for $i \in \{1, \ldots, n-1\}$, and $y = b - a_0$, and $\xi' = \xi_{\sigma^n}$. We define the number $t(\tau^{n-1}, \sigma^n, \mathbb{R}^m, \xi)$ to be 1 if

$$\det\begin{pmatrix} x_1 \cdot x_1 & \cdots & x_{n-1} \cdot x_1 & h_{\xi'}(x_1) \\ \vdots & & \vdots & \vdots \\ x_1 \cdot x_{n-1} & \cdots & x_{n-1} \cdot x_{n-1} & h_{\xi'}(x_{n-1}) \\ x_1 \cdot y & \cdots & x_{n-1} \cdot y & h_{\xi'}(y) \end{pmatrix} > 0, \tag{1}$$

and we define $t(\tau^{n-1}, \sigma^n, \mathbb{R}^m, \xi)$ to be 0 otherwise.

The following lemma gives us an intuitive picture of what the above definition means. We use the following notation: if x is a real number, let sgn x be -1, 0, 1, respectively if x is negative, zero or positive, respectively.

LEMMA 3.6. *Let K be an n-dimensional simplicial complex in $\mathbb{R}^m$, let $\sigma^n = \langle a_0, \ldots, a_{n-1}, b\rangle$ be an n-simplex of K, let $\tau^{n-1} = \langle a_0, \ldots, a_{n-1}\rangle$, and let $\xi \in S^{m-1}$ be an allowable vector with respect to K. We use the abbreviations $x_i = a_i - a_0$ for $i \in \{1, \ldots, n-1\}$, and $y = b - a_0$, and $\xi' = \xi_{\sigma^n}$. By translation we can think of σ^n as sitting in $V(\sigma^n)$. The following are equivalent.*

(1) $t(\tau^{n-1}, \sigma^n, \mathbb{R}^m, \xi) = 1$.
(2) $\operatorname{sgn}\det(x_1|\cdots|x_{n-1}|y) = \operatorname{sgn}\det\big(x_1|\cdots|\ x_{n-1}|\xi'\big)$,
where we think of $x_1, \ldots, x_{n-1}, y, \xi'$ as column vectors in $\mathbb{R}^n$.
(3) *If t is a point in the relative interior of τ^{n-1}, and if s is a point in the relative interior of σ^n such that the vector $s - t$ is orthogonal to $V(\tau^{n-1})$, then $h_\xi(s) > h_\xi(t)$.*

Proof. We can translate all of $\mathbb{R}^m$ so that a_0 is taken to the origin; hence we will think of a_0 as equaling 0.

Using the basis $\{x_1, \ldots, x_{n-1}, y\}$ for $V(\sigma^n)$, we can write

$$\xi' = c_1x_1 + \cdots + c_{n-1}x_{n-1} + py, \tag{2}$$

for some real numbers $c_1, \ldots, c_{n-1}, p$. By the definition of ξ being an allowable vector, it follows that $p \neq 0$. We will show that each of Conditions (1)–(3) holds iff $p > 0$, and that will prove that the three conditions are equivalent.

To show that Condition (1) holds iff $p > 0$, we solve Equation (2) for p by taking the inner product of it with each of $x_1, \ldots, x_{n-1}, y$, and then solving the resulting system of linear equations using Cramer's rule, to obtain

$$p = \frac{\det\begin{pmatrix} x_1\cdot x_1 & \cdots & x_{n-1}\cdot x_1 & \xi'\cdot x_1 \\ \vdots & & \vdots & \vdots \\ x_1\cdot x_{n-1} & \cdots & x_{n-1}\cdot x_{n-1} & \xi'\cdot x_{n-1} \\ x_1\cdot y & \cdots & x_{n-1}\cdot y & \xi'\cdot y \end{pmatrix}}{\det\begin{pmatrix} x_1\cdot x_1 & \cdots & x_{n-1}\cdot x_1 & y\cdot x_1 \\ \vdots & & \vdots & \vdots \\ x_1\cdot x_{n-1} & \cdots & x_{n-1}\cdot x_{n-1} & y\cdot x_{n-1} \\ x_1\cdot y & \cdots & x_{n-1}\cdot y & y\cdot y \end{pmatrix}}. \tag{3}$$

Next, observe that the matrix in the denominator of the right-hand side of Equation (3) can be written as

$$(x_1|\cdots|\ x_{n-1}|y)^T(x_1|\cdots|\ x_{n-1}\,|\,y). \tag{4}$$

It follows that the denominator in the right hand side of Equation (3) is always positive. We deduce that $p > 0$ iff the numerator in Equation (3) is positive. As seen above, we know that $\xi = a\xi' + w$, where a is some positive constant, and where w is a vector that is orthogonal to $V(\sigma^n)$, and it follows that $h_\xi(v) = v \cdot \xi = a(v \cdot \xi')$ for any vector $v \in V(\sigma^n)$. It is then straightforward to deduce that the numerator in the right-hand side of Equation (3) is positive iff Equation (1) holds. It follows that Condition (1) holds iff $p > 0$.

To show that Condition (2) holds iff $p > 0$, we use Equation (2) and basic properties of determinants, to see that

$$\det\big(x_1|\cdots|\ x_{n-1}\,|\,\xi'\big) = p\det(x_1|\cdots|x_{n-1}\,|\,y). \tag{5}$$

Equation (5) clearly shows that Condition (2) holds iff $p > 0$.

We now show that Condition (3) holds iff $p > 0$. Let t be a point in the relative interior of τ^{n-1}, and let s is a point in the relative interior of σ^n such that the vector $s - t$ is orthogonal to $V(\tau^{n-1})$. We need to show that $h_\xi(s) > h_\xi(t)$ iff $p > 0$. Let $z = s - t$. It will suffice to show that $h_\xi(z) > 0$ iff $p > 0$. We observe that $\xi = a\xi' + w$, where a is some positive constant, and where w is a vector that is orthogonal to $V(\sigma^n)$. Because $z \in V(\sigma^n)$, it follows that $h_\xi(z) = z \cdot \xi = a(z \cdot \xi') = ah_{\xi'}(z)$. Hence, it will suffice to show that $h_{\xi'}(z) > 0$ iff $p > 0$.

Because s is a point in the relative interior of σ^n, it follows (using barycentric coordinates) that

$$s = d_0a_0 + \cdots + d_{n-1}a_{n-1} + eb, \tag{6}$$

for some real numbers $d_0, \ldots, d_{n-1}, e$, where $d_0, \ldots, d_{n-1}, e > 0$. Similarly, Because t is a point in the relative interior of $V(\tau^{n-1})$, it follows that

$$t = f_0a_0 + \cdots + f_{n-1}\ a_{n-1}, \tag{7}$$

for some real numbers $f_0, \ldots, f_{n-1}$. Recall that we are assuming that $a_0 = 0$, and hence $x_i = a_i$ for $i \in \{1, \ldots, n-1\}$, and $y = b$. Combining Equations (6) and (7) we see that

$$z = (d_1 - f_1)x_1 + \cdots + (d_{n-1} - f_{n-1})x_{n-1} + ey. \tag{8}$$

Next, using the basis $\{x_1, \ldots, x_{n-1}, z\}$ for $V(\sigma^n)$, we can write

$$\xi' = k_1x_1 + \cdots + k_{n-1}\ x_{n-1} + rz, \tag{9}$$

for some real numbers $k_1, \ldots, k_{n-1}, r$, where $r \neq 0$. Because z is orthogonal to $x_1, \ldots, x_{n-1}$, it follows that $z \cdot \xi' = r|z|^2$. It follows that $z \cdot \xi' > 0$ iff $r > 0$, which is equivalent to saying that $h_{\xi'}(z) > 0$ iff $r > 0$. Next, we substitute Equation (8) into (9) and rearrange to obtain

$$\xi' = u_1x_1 + \cdots + u_{n-1}x_{n-1} + rey, \tag{10}$$

for appropriate real numbers $u_1, \ldots, u_{n-1}$. Comparing Equations (10) and (2), we deduce that $re = p$. Because $e > 0$, we see that $p > 0$ iff $r > 0$. Having already seen that $h_{\xi'}(z) > 0$ iff $r > 0$, it follows that $h_{\xi'}(z) > 0$ iff $p > 0$, which is what we needed to show regarding Condition (3). $\square$

Remark 3.7. We see from the definition of $t(\tau^{n-1}, \sigma^n, \mathbb{R}^m, \xi)$ that $t(\tau^{n-1}, \sigma^n, \mathbb{R}^m, \xi) = 1$ iff ξ_{σ^n} points across τ^{n-1} in the direction of σ^n.

Our next definition is as follows.

DEFINITION 3.8. Let K be an n-dimensional simplicial complex in $\mathbb{R}^m$, let σ^n be an n-simplex of K, let η^i be an i-face of σ^n with $0 \leqslant i \leqslant n-2$ and let $\xi \in S^{m-1}$ be an allowable vector with respect to K. We define the number $g(\eta^i, \sigma^n, \mathbb{R}^m, \xi)$ as follows. The simplex η^i is the intersection of precisely $n-i$ $(n-1)$-faces of σ^n, say $\tau_1, \ldots, \tau_{n-i}$. Then $g(\eta^i, \sigma^n, \mathbb{R}^m, \xi)$ is defined by

$$g(\eta^i, \sigma^n, \mathbb{R}^m, \xi) = \prod_{k=1}^{n-i} t(\tau_k, \sigma^n, \mathbb{R}^m, \xi) + \prod_{k=1}^{n-i} t(\tau_k, \sigma^n, \mathbb{R}^m, -\xi).$$

Remark 3.9. It is seen from the above definition that $g(\eta^i, \sigma^n, \mathbb{R}^m, \xi) = 1$ iff $t(\tau_k, \sigma^n, \mathbb{R}^m, \xi) = 1$ for all $k \in \{1, \ldots, n-i\}$, or if $t(\tau_k, \sigma^n, \mathbb{R}^m, -\xi) = 1$ for all $k \in \{1, \ldots, n-i\}$; and $g(\eta^i, \sigma^n, \mathbb{R}^m, \xi) = 0$ otherwise. This means is that $g(\eta^i, \sigma^n, \mathbb{R}^m, \xi) = 1$ iff either ξ_{σ^n} or $-\xi_{\sigma^n}$ points inside the angle in σ^n along η^i.

Finally, we can now give the definition of the index of simplices with respect to a projection map.

DEFINITION 3.10. Let K be an n-dimensional simplicial complex in $\mathbb{R}^m$, let η^i be an i-simplex of K with $0 \leqslant i \leqslant n-2$, and let $\xi \in S^{m-1}$ be an allowable vector with respect to K. The *index* of η^i with respect to ξ is defined to be the number $i(\eta^i, \xi)$ given by

$$i(\eta^i, \xi) = T_n(\eta^i) - \frac{1}{2} \sum_{\sigma^n \succ \eta^i} g(\eta^i, \sigma^n, \mathbb{R}^m, \xi), \tag{11}$$

where the summation is over all n-simplices σ^n which have η^i as a face.

EXAMPLE 3.11. We continue Example 2.3. Let ξ be as shown in Figure 4, where we have assumed that the vertex w is at the origin. It is seen that $g(a, \sigma_1, \mathbb{R}^2, \xi) = g(d, \sigma_2, \mathbb{R}^2, \xi) = g(e, \sigma_3, \mathbb{R}^2, \xi) = 1$, and that all other relevant numbers of the form $g(x, \sigma_i, \mathbb{R}^2, \xi)$ are 0. Hence, we compute

$$i(w, \xi) = T_2(w) - \frac{1}{2} \sum_{i=1}^{3} g(w, \sigma_i, \mathbb{R}^2, \xi) = 1 - \tfrac{1}{2} \cdot 0 = 1,$$

and

$$i(a, \xi) = T_2(a) - \tfrac{1}{2} g(a, \sigma_1, \mathbb{R}^2, \xi) = \tfrac{1}{2} - \tfrac{1}{2} \cdot 1 = 0;$$

it is similarly seen that

$$i(b, \xi) = i(c, \xi) = i(f, \xi) = \tfrac{1}{2} \quad \text{and} \quad i(d, \xi) = i(e, \xi) = 0.$$

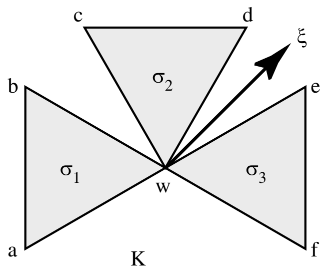

Figure 4.

Observe that the sum of the indices at the vertices of K is $5/2$, which equals $\chi^s(K)$, as computed in Example 2.9. We will see in Theorem 3.12 below that this equality is not coincidental.

By way of comparison, Banchoff's index at the vertices of K (as defined in [1, Section 1]) is index 1 at the vertex a, and index 0 at all the other vertices. The sum of Banchoff's indices at the vertices of K is 1, which equals $\chi(K)$, as expected. (In the case of two-dimensional simplicial complexes it is plausible to compare Banchoff's index at a vertex with respect to a projection map with the index as we have defined it with respect to the same projection map; the above example shows that the two different types of indices are in general different. In the case of higher dimensional simplicial complexes, it is difficult to compare the two types of indices, because Banchoff's is defined only at the vertices, whereas our index is defined at all simplices up to co-dimension 2.)

We can now state our main results; their proofs will be given in Section 4. Our first theorem concerns the sum of the indices of the simplices of co-dimension at least 2 of a simplicial complex. This theorem is the analog of Theorem 1 in [1, p. 247], which he calls the Critical Point Theorem.

THEOREM 3.12. *Let K be an n-dimensional simplicial complex in $\mathbb{R}^m$, and let $\xi \in S^{m-1}$ be an allowable vector with respect to K. Then*

$$\sum_{\substack{\eta^i \in K \\ 0 \leqslant i \leqslant n-2}} (-1)^i i(\eta^i, \xi) = \chi^s(K).$$

Our second theorem is the analog of Theorem 3 in [1, p. 251], which he calls the Theorema Egregium. Let $d\omega^{m-1}$ denote the ordinary volume element on S^{m-1}. Observe that if K is an n-dimensional simplicial complex in $\mathbb{R}^m$, and if η^i is an i-simplex of K with $0 \leqslant i \leqslant n-2$, then by Lemma 3.2, the set of $\xi \in S^{m-1}$ for which $i(\eta^i, \xi)$ is defined is an open dense subset of S^{m-1}. Hence, it is possible to integrate $i(\eta^i, \xi)$ over S^{m-1}. Recall that we adopt the convention that all angles are normalized so that the volume of the unit $(n-1)$-sphere in $(n-1)$-measure is 1 in all dimensions.

THEOREM 3.13. *Let K be an n-dimensional simplicial complex in $\mathbb{R}^m$, and let η^i be an i-simplex of K with $0 \leqslant i \leqslant n-2$. Then*

$$\int_{S^{m-1}} i(\eta^i, \xi)\, \mathrm{d}\omega^{m-1} = D_n(\eta^i).$$

4. Proofs

We start with the proof of Theorem 3.12, the essence of which is the following lemma. The proof of this lemma uses the main idea of the proof of Gram's Theorem given in [24, pp. 22–24]; Hopf attributes his proof to Poincaré. See [14, Section 14.4, 15] for more about Gram's Theorem and its generalization to convex polytopes. Note that [16] refers to this result at the Gram–Euler Theorem, and [25, p. 174] refers to it as the Brianchon–Gram Theorem (Hopf does not give it any name).

LEMMA 4.1. *Let K be an n-dimensional simplicial complex in $\mathbb{R}^m$, let σ^n be an n-simplex of K, and let $\xi \in S^{m-1}$ be an allowable vector with respect to K. Then*

$$\sum_{\substack{\eta^i \prec \sigma^n \\ 0 \leqslant i \leqslant n-2}} (-1)^i g(\eta^i, \sigma^n, \mathbb{R}^m, \xi) = (-1)^n (n-1).$$

Proof. We can think of σ^n as sitting in $V(\sigma^n)$ by translation, we can think of $V(\sigma^n)$ as identified with $\mathbb{R}^n$, and we can think of σ^n as an n-dimensional simplicial complex. It is straightforward to see that for each $(n-1)$-face τ^{n-1} of σ^n, we have $t(\tau^{n-1}, \sigma^n, \mathbb{R}^m, \xi) = t(\tau^{n-1}, \sigma^n, \mathbb{R}^n, \xi_{\sigma^n})$. It follows that for each i-face η^i of σ^n with $0 \leqslant i \leqslant n-2$, we have $g(\eta^i, \sigma^n, \mathbb{R}^m, \xi) = g(\eta^i, \sigma^n, \mathbb{R}^n, \xi_{\sigma^n})$. It therefore suffices to show that

$$\sum_{\substack{\eta^i \prec \sigma^n \\ 0 \leqslant i \leqslant n-2}} (-1)^i g(\eta^i, \sigma^n, \mathbb{R}^n, \xi_{\sigma^n}) = (-1)^n (n-1). \tag{12}$$

Hence, we will work entirely in $\mathbb{R}^n$ for the rest of the proof.

As stated in Remark 3.4, we know that ξ_{σ^n} is an allowable vector in S^{n-1} with respect to σ^n. Let $\tau_0, \ldots, \tau_n$ denote the $(n-1)$-faces of σ^n. We use the abbreviations $\xi' = \xi_{\sigma^n}$, and $F_i(\xi') = t(\tau_i, \sigma^n, \mathbb{R}^n, x)$, where $i \in \{0, \ldots, n\}$ and $x \in S^{n-1}$.

It is trivial to see that for each $i \in \{0, \ldots, n\}$, we have

$$F_i(\xi') + F_i(-\xi') = 1. \tag{13}$$

Now, suppose that $p > 1$, and that $0 \leqslant i_1 < i_2 < \cdots < i_p \leqslant n$. Observe that $\tau_{i_1} \cap \cdots \cap \tau_{i_p}$ is an $(n-p)$-face of σ^n. Then, by the definition of $g(\tau_{i_1} \cap \cdots \cap \tau_{i_p}, \sigma^n, \mathbb{R}^n, \xi')$, we have

$$\begin{aligned} &F_{i_1}(\xi') F_{i_2}(\xi') \cdots F_{i_p}(\xi') + F_{i_1}(-\xi') F_{i_2}(-\xi') \cdots F_{i_p}(-\xi') \\ &\quad = g(\tau_{i_1} \cap \cdots \cap \tau_{i_p}, \sigma^n, \mathbb{R}^n, \xi'). \end{aligned} \tag{14}$$

As noted in [24, p. 24], it is seen that

$$\prod_{i=0}^{n} F_i(\xi') = 0 \quad \text{and} \quad \prod_{i=0}^{n}(1 - F_i(\xi')) = 0. \tag{15}$$

By expanding the second part of Equation (15), and then using the first part of the equation, we deduce that

$$\begin{aligned} 1 - \sum_{0 \leqslant i \leqslant n} F_i(\xi') + \sum_{0 \leqslant i < j \leqslant n} F_i(\xi')F_j(\xi') - \cdots \\ + (-1)^n \sum_{0 \leqslant i_1 < i_2 < \cdots < i_n \leqslant n} F_{i_1}(\xi')F_{i_2}(\xi') \cdots F_{i_n}(\xi') = 0. \end{aligned} \tag{16}$$

We can substitute $-\xi'$ for ξ' into Equation (16), and then add the resulting equation to Equation (16), which yields

$$\begin{aligned} &2 - \sum_{0 \leqslant i \leqslant n} [F_i(\xi') + F_i(-\xi')] + \sum_{0 \leqslant i < j \leqslant n} [F_i(\xi')F_j(\xi') + F_i(-\xi')F_j(-\xi')] - \cdots \\ &+ (-1)^n \sum_{0 \leqslant i_1 < i_2 < \cdots < i_n \leqslant n} [F_{i_1}(\xi')F_{i_2}(\xi') \cdots F_{i_n}(\xi') + F_{i_1}(-\xi')F_{i_2}(-\xi') \cdots F_{i_n}(-\xi')] \\ &= 0. \end{aligned} \tag{17}$$

Next, substituting Equations (13) and (14) into Equation (17), we obtain

$$\begin{aligned} 2 - \sum_{0 \leqslant i \leqslant n} 1 + \sum_{0 \leqslant i < j \leqslant n} g(\tau_i \cap \tau_j, \sigma^n, \mathbb{R}^n, \xi') - \cdots \\ + (-1)^n \sum_{0 \leqslant i_1 < i_2 < \cdots < i_n \leqslant n} g(\tau_{i_1} \cap \cdots \cap \tau_{i_n}, \sigma^n, \mathbb{R}^n, \xi') = 0. \end{aligned} \tag{18}$$

For each $p > 1$, we observe that the collection of all intersections of the form $\tau_{i_1} \cap \cdots \cap \tau_{i_p}$ precisely equals the collection of all $(n-p)$-faces of σ^n. Hence, Equation (18) can be rewritten as

$$2 - (n+1) + \sum_{\eta^{n-2} \prec \sigma^n} g(\eta^{n-2}, \sigma^n, \mathbb{R}^n, \xi') - \cdots + (-1)^n \sum_{\eta^0 \prec \sigma^n} g(\eta^0, \sigma^n, \mathbb{R}^n, \xi') = 0, \tag{19}$$

where the each of the summations is over all the faces of σ^n of the appropriate dimension. Rearranging Equation (19) yields Equation (12), which is what we needed to show. □

Proof of Theorem 3.12. We will need to use the following equation, which was given in [11, p. 387]:

$$\begin{aligned} &\sum_{\sigma^n \in K} \frac{(-1)^n(n-1)}{2} \\ &\quad = -\sum_{\sigma^{n-1} \in K} T_n(\sigma^{n-1})(-1)^{\dim \sigma^{n-1}} - \sum_{\sigma^n \in K} T_n(\sigma^n)(-1)^{\dim \sigma^n}. \end{aligned} \tag{20}$$

We now compute

$$\begin{aligned}
\sum_{\substack{\eta^i \in K \\ 0 \leqslant i \leqslant n-2}} (-1)^i i(\eta^i, \xi) &= \sum_{\substack{\eta^i \in K \\ 0 \leqslant i \leqslant n-2}} (-1)^i \left[T_n(\eta^i) - \frac{1}{2} \sum_{\sigma^n \succ \eta^i} g(\eta^i, \sigma^n, \mathbb{R}^m, \xi) \right] \\
&= \sum_{\substack{\eta^i \in K \\ 0 \leqslant i \leqslant n-2}} (-1)^i T_n(\eta^i) - \sum_{\substack{\eta^i \in K \\ 0 \leqslant i \leqslant n-2}} (-1)^i \frac{1}{2} \sum_{\sigma^n \succ \eta^i} g(\eta^i, \sigma^n, \mathbb{R}^m, \xi) \\
&= \sum_{\substack{\eta^i \in K \\ 0 \leqslant i \leqslant n-2}} T_n(\eta^i)(-1)^{\dim \eta^i} - \frac{1}{2} \sum_{\sigma^n \in K} \sum_{\substack{\eta^i \prec \sigma^n \\ 0 \leqslant i \leqslant n-2}} (-1)^i g(\eta^i, \sigma^n, \mathbb{R}^m, \xi) \\
&= \sum_{\substack{\eta^i \in K \\ 0 \leqslant i \leqslant n-2}} T_n(\eta^i)(-1)^{\dim \eta^i} - \frac{1}{2} \sum_{\sigma^n \in K} (-1)^n (n-1) \\
&\qquad \text{using Lemma 4.1} \\
&= \sum_{\substack{\eta^i \in K \\ 0 \leqslant i \leqslant n-2}} T_n(\eta^i)(-1)^{\dim \eta^i} + \sum_{\sigma^{n-1} \in K} T_n(\sigma^{n-1})(-1)^{\dim \sigma^{n-1}} + \\
&\qquad + \sum_{\sigma^n \in K} T_n(\sigma^n)(-1)^{\dim \sigma^n} \\
&\qquad \text{using Equation (20)} \\
&= \sum_{\eta \in K} T_n(\eta)\,(-1)^{\dim \eta} = \chi^s(K),
\end{aligned}$$

where the last equality is by the definition of $\chi^s(K)$. □

We now turn to the Proof of Theorem 3.13. We start with the following lemma, which relates $g(\eta^i, \sigma^n, \mathbb{R}^m, \xi)$ to the interior angle of σ^n along η^i.

LEMMA 4.2. *Let K be an n-dimensional simplicial complex in $\mathbb{R}^m$, let σ^n be an n-simplex of K, and let η^i be an i-face of σ^n with $0 \leqslant i \leqslant n-2$. Then*

$$\frac{1}{2} \int_{S^{m-1}} g(\eta^i, \sigma^n, \mathbb{R}^m, \xi)\, \mathrm{d}\omega^{m-1} = \alpha(\eta^i, \sigma^n).$$

Proof. We can think of σ^n as sitting in $V(\sigma^n)$ by translation, and we can think of $V(\sigma^n)$ as identified with $\mathbb{R}^n$. Additionally, we can think of σ^n as an n-dimensional simplicial complex.

Let $\zeta \in S^{n-1}$ be an allowable vector with respect to σ^n. The simplex η^i is in the intersection of precisely $n-i$ $(n-1)$-faces of σ^n, say $\tau_1, \ldots, \tau_{n-i}$. As stated in Remark 3.9, we see that $g(\eta^i, \sigma^n, \mathbb{R}^n, \zeta) = 1$ iff either ζ or $-\zeta$ points inside the angle in σ^n along η^i. It now follows easily that

$$\frac{1}{2} \int_{S^{n-1}} g(\eta^i, \sigma^n, \mathbb{R}^n, \zeta)\ \mathrm{d}\omega^{n-1} = \alpha(\eta^i, \sigma^n). \tag{21}$$

Next, we assert that

$$\int_{S^{m-1}} g(\eta^i, \sigma^n, \mathbb{R}^m, \xi)\, \mathrm{d}\omega^{m-1} = \int_{S^{n-1}} g(\eta^i, \sigma^n, \mathbb{R}^n, \zeta)\, \mathrm{d}\omega^{n-1}. \tag{22}$$

The lemma follows by combining Equations (21) and (22). Equation (22) can be proved similarly to the argument given in the proof of Lemma 2 of [1]; we omit the details. (We note that Equation (22) is true as stated only because we are assuming that the volume of the unit $(k-1)$-sphere in $(k-1)$-measure is 1 in all dimensions; otherwise we would need factors consisting of the volumes of appropriate unit spheres.) $\square$

Proof of Theorem 3.13. First, observe that we have

$$\int_{S^{m-1}} \mathrm{d}\omega^{m-1} = 1, \tag{23}$$

given our normalization of the volumes of unit spheres.

We then compute

$$\begin{aligned}
&\int_{S^{m-1}} i(\eta^i, \xi)\, \mathrm{d}\omega^{m-1} = \\
&\quad = \int_{S^{m-1}} \left[T_n(\eta^i) - \frac{1}{2} \sum_{\sigma^n \succ \eta^i} g(\eta^i, \sigma^n, \mathbb{R}^m, \xi) \right] \mathrm{d}\omega^{m-1} \\
&\quad = T_n(\eta^i) \int_{S^{m-1}} \mathrm{d}\omega^{m-1} - \sum_{\sigma^n \succ \eta^i} \frac{1}{2} \int_{S^{m-1}} g(\eta^i, \sigma^n, \mathbb{R}^m, \xi)\, \mathrm{d}\omega^{m-1} \\
&\quad = T_n(\eta^i) - \sum_{\sigma^n \succ \eta^i} \alpha(\eta^i, \sigma^n) = D_n(\eta^i),
\end{aligned}$$

where the last equality uses Equation (23), Lemma 4.2 and the definition of $D_n(\eta^i)$. $\square$

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