A Simple Proof of a Generalized No Retraction Theorem

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1. INTRODUCTION. The world does not need yet another proof of the classical no retraction theorem (NRT) and its equivalent partner the Brouwer fixed point theorem (BFPT)—many lovely elementary proofs are widely known. What does merit a new proof, however, is a much less well-known generalization of the NRT to a broader class of topological spaces than only those that are homeomorphic to balls (which is what the NRT and BFPT are traditionally about).

The purpose of this paper is to state and prove this generalized NRT in the 2-dimensional case. In particular, we will show that a version of the NRT, when appropriately stated, can be proved for the class of all topological spaces that are homeomorphic to the underlying spaces of finite 2-dimensional simplicial complexes. (As the reader can verify by finding examples, the BFPT does not generalize to all such spaces, and hence the equivalence of the NRT and BFPT also does not generalize to all such spaces.) If one were interested only in the classical NRT, our proof could be simplified even further to give a particularly low-tech proof of that theorem, though we omit the details.

To remind the reader of what we are generalizing, we state the classical 2-dimensional versions of the NRT and the BFPT. The exact analogs of both theorems hold in higher dimensions, though we will not discuss them in this note. We use the notations D^2 and S^1 to denote the unit ball and the unit circle respectively in $\mathbb{R}^2$; that is,

$$D^2 = \{x \in \mathbb{R}^2 \mid \|x\| \leq 1\} \qquad \text{and} \qquad S^1 = \{x \in \mathbb{R}^2 \mid \|x\| = 1\}.$$

We think of S^1 as the boundary of D^2. Also, if X is a topological space and $A \subseteq X$, a **retraction** $r\colon X \to A$ is a continuous map such that $r(a) = a$ for all $a \in A$.

Theorem 1.1 (No Retraction Theorem). *There is no retraction $r\colon D^2 \to S^1$.*

Theorem 1.2 (Brouwer Fixed Point Theorem). *Let $f\colon D^2 \to D^2$ be a continuous map. Then f has a fixed point.*

The proof that the NRT and the BFPT are equivalent is simple, and can be found in many texts on topology; see for example [**5**, pp. 272–273]. The most common way to prove these theorems is to prove the NRT first, and then to deduce the BFPT from the NRT. A widely used short proof of the NRT relies upon algebraic topology—the fundamental group in the 2-dimensional case, and homology groups in higher dimensions; see for example [**9**, p. 348] for the 2-dimensional case, and [**8**, p. 117] for the higher-dimensional case.

There are also a number of very nice elementary proofs of the NRT that do not involve algebraic topology, for example the well-known proof found in [**7**], with a variant found in [**10**]. Another well-known proof of the NRT is given in [**4**]. This proof is very short and clear, and also avoids algebraic topology, though it uses the simplicial approximation theorem, which makes it not quite as low-tech as it at first appears. Moreover, it has been pointed out recently in [**6**] that the standard simplicial approximation theorem does not suffice for [**4**], and that the relative simplicial approximation theorem of [**15**] is needed, thereby diminishing somewhat the elementary nature of this proof.

Although all the ingredients of the generalized NRT (Theorem 3.1 below) and its proof are well-known, strangely the statement of this theorem does not appear to be widely known. Indeed, the only place that the author has found the full version of the generalized NRT is [**12**]; the proof in that paper is short and clever, and it avoids both algebraic topology and the simplicial approximation theorem, though its use of induction over the dimensions of the spaces involved makes the proof not entirely transparent. The author has found two published versions of the generalized NRT that are not as general as possible, one in [**11**], where there is a short and clever proof of the generalized NRT for simplicial complexes that have boundary a sphere (the proof has the same merits and drawbacks as [**12**]), and another in [**14**, pp. 150–151], where there is an outline of a proof (given as exercises) of the generalized NRT restricted to pseudomanifolds with boundary (the proof uses the simplicial approximation theorem).

The purpose of the present note is both to bring the generalized NRT to the attention of a wider audience, and to give an even more elementary and transparent proof of the generalized NRT than the proofs given in [**12**], [**11**], and [**14**, pp. 150–151]. Another advantage of our method is that the NRT can be generalized without much more effort even further than we present here (and than is found in the three references just cited), using the notion of mod 2 degree; see [**1**] for full details in all dimensions.

The tools in our proof involve nothing more technical than finite 2-dimensional simplicial complexes in Euclidean space, subdivision and simplicial maps, and the basic definition of a graph (in the sense of graph theory). As a very brief reminder, it suffices for our purposes to think of a 2-dimensional simplicial complex as a finite collection of triangles in some Euclidean space, such that if two triangles have nonempty intersection, they intersect in either a common vertex or a common edge; see Figure 1. (Technically, a simplicial complex also contains as separate simplices all the edges and vertices of all the triangles, but we will not refer to that.) If K is a 2-dimensional simplicial complex, the underlying space of K, denoted $|K|$, is the union of the triangles in K; a subdivision of K is a new 2-dimensional simplicial complex that results from breaking up the triangles of K into smaller triangles, while maintaining proper intersections between the new triangles. A simplicial map from one 2-dimensional simplicial complex to another is a continuous map that takes each triangle of the domain affine linearly onto a triangle, edge, or vertex of the codomain. A graph, for our purposes, is just the 1-dimensional analog of a 2-dimensional simplicial complex, that is, a finite collection of edges in some Euclidean space, such that if two edges have nonempty intersection, they intersect in a common vertex; see Figure 2.

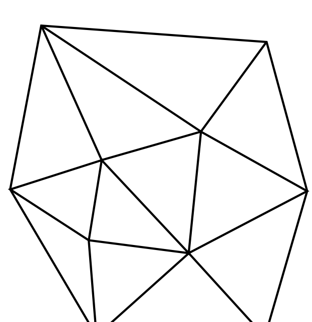

Figure 1.

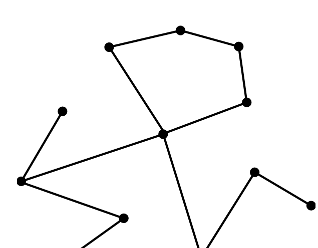

Figure 2.

The NRT will be seen to be an issue of parity. The key to our proof is the following elementary result from graph theory, the proof of which is sufficiently simple to be left to the reader; a proof, and further discussion, can be found in many introductory texts on graph theory, for example [**3**, p. 14]. This lemma could fairly be said to be the core of the NRT.

Lemma 1.3. *Let G be a graph with no loops. The number of vertices of G that are contained in odd numbers of edges is even.*

2. WHAT IS A BOUNDARY? The standard NRT is about continuous maps from the unit ball to its boundary. The theorem also holds if the unit ball is replaced by anything homeomorphic to it.

The generalized NRT is about continuous maps from a larger class of topological spaces to their boundaries. The statement and proof of the generalized NRT will be given in Section 3, but before we can state the theorem, we need to discuss its most interesting aspect, which is finding the right definition of boundary in this generalized context.

The term "boundary" in topology has a number of meanings. In arbitrary topological spaces, the boundary of a set is the intersection of its closure and the closure of its complement. For manifolds with boundary, there is a different definition of boundary. In the case of a disk in the plane, these two notions of boundary coincide, though they do not coincide in general. The notion of boundary that is relevant here is the boundary of a manifold with boundary. In the 2-dimensional case, a manifold with boundary is just a surface with boundary, which we now discuss informally.

The model of the boundary of a surface with boundary is the boundary of D^2, which is S^1. No fancy definition is needed for the boundary of D^2, and all that is used is the notion of distance in $\mathbb{R}^2$.

Next, suppose that X is a topological space that is homeomorphic to D^2. It can be proved, using invariance of domain, that if $f : D^2 \to X$ is a homeomorphism, then $f(S^1)$ is independent of the choice of the map f, and hence the boundary of X is defined to be $f(S^1)$ for any choice of homeomorphism f. See [**2**, pp. 50–51] for a proof of this fact. See [**9**, pp. 383–384] for a proof of the 2-dimensional version of invariance of domain using the fundamental group, and see [**8**, p. 207] for a proof of invariance of domain in all dimensions using homology.

More generally, invariance of domain can be used to define the boundary of any surface with boundary. See Figure 3 for an example of a surface with boundary. We will not provide a rigorous treatment of surfaces with boundary here, because we are interested in surfaces with boundary only for the purposes of analogy. However, we note one very important property of the boundary of a surface with boundary, which is

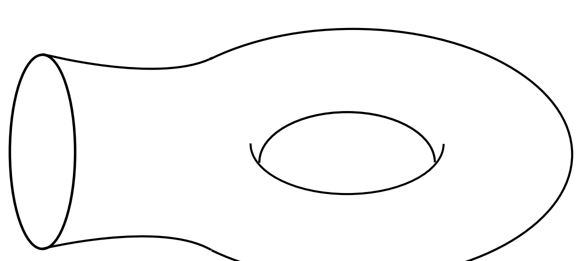

Figure 3.

that the boundary, viewed as a 1-dimensional manifold, does not itself have a boundary. The general idea that the boundary of a boundary is empty is very important in various parts of mathematics, for example in homology theory, and we will return to this idea shortly.

For our generalized NRT, we want to consider spaces that might not be surfaces with boundary. For example, we see in Figure 4 the result of taking three disks, gluing their boundaries along a common circle (the middle disk is shaded, and the top and bottom disks are transparent), removing two small disks from the middle disk and one small disk from each of the top and bottom disks, and adding tubes where the small disks were removed as shown in the figure. This object is not a surface (with or without boundary), though it is certainly a topological space. We will shortly assign to this object a "boundary," in a way that generalizes, and is consistent with, the notion of boundary for surfaces with boundary. The key observation is that the object shown in Figure 4 is homeomorphic to the underlying space of a finite 2-dimensional simplicial complex, and we will use this simplicial structure to help us find the boundary.

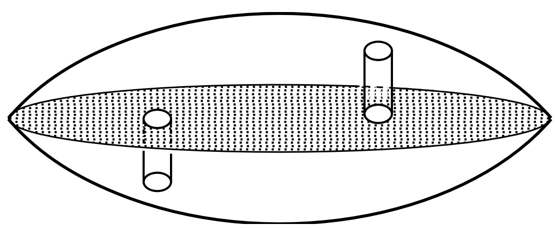

Figure 4.

We use the term "simplicial surface" to denote any 2-dimensional simplicial complex with underlying space that is a surface without boundary. For example, the octahedron shown in Figure 5 is a simplicial surface. It is immediately observed that every edge in the octahedron is contained in precisely two triangles, and it might be tempting to conclude that simplicial surfaces are precisely those 2-dimensional simplicial complexes in which every edge is contained in two triangles, but that would be incorrect. For example, the 2-dimensional simplicial complex shown in Figure 6 satisfies this condition, though its underlying space is certainly not a surface. It is standard to call a 2-dimensional simplicial complex K a **2-dimensional pseudomanifold** if every edge of K is contained in two triangles. (Actually, the common definition of pseudomanifold also contains two other conditions, one concerning pure 2-dimensionality and the other concerning connectivity, but we will not need these conditions for our discussion; see for example [**5**, p. 252] for more about pseudomanifolds.)

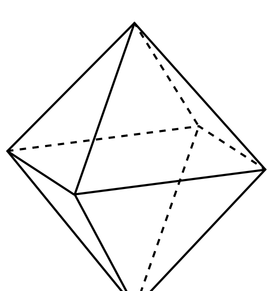

Figure 5.

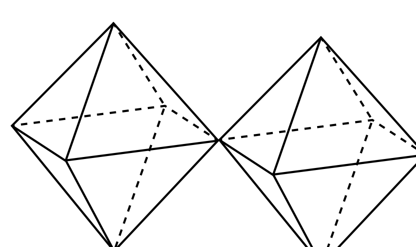

Figure 6.

What about simplicial surfaces with boundary? Consider a triangulated disk, for example the one shown in Figure 1. It is clear that the boundary of the disk consists of those edges that are contained in exactly one triangle each. The edges in the interior of the disk are contained in two triangles each. It is standard to call a 2-dimensional simplicial complex K a **2-dimensional pseudomanifold with boundary** if every edge of K is contained in one or two triangles; the **boundary** of K, denoted $\text{Bd}\, K$, is the collection of all edges of K that are contained in one triangle each. (Again, the common definition contains two additional conditions that we will not need.) For example, the 2-dimensional simplicial complex shown in Figure 7 is a 2-dimensional pseudomanifold with boundary; its boundary is a figure eight.

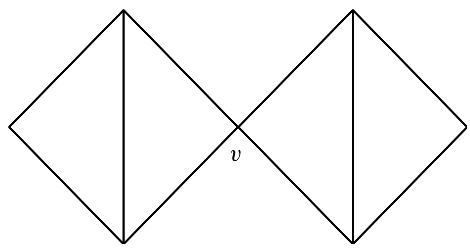

Figure 7.

So far so good, but there is a problem with the above definition. Recall the idea, stated above for surfaces with boundary, that the boundary of a boundary should be empty. Does that hold for 2-dimensional pseudomanifolds with boundary? Suppose that K is a 2-dimensional pseudomanifold with boundary. It is evident that $\text{Bd}\, K$ is 1-dimensional. To say that the boundary of the boundary of K is empty ought to mean that $\text{Bd}\, K$ is a 1-dimensional pseudomanifold *without boundary*, which means that every vertex in $\text{Bd}\, K$ is contained in precisely two edges. For example, if we take K to be the disk shown in Figure 1, then clearly $\text{Bd}\, K$ is a circle, and every vertex in $\text{Bd}\, K$ is indeed contained in two edges, as we would hope. However, if we take K to be the 2-dimensional simplicial complex shown in Figure 7, then $\text{Bd}\, K$ is a figure eight, and the vertex v is contained in four edges, not two. Hence, it is not the case that $\text{Bd}\, K$ is always a 1-dimensional pseudomanifold.

As is often the case in mathematics, when things don't work out as expected, we can try to save the situation by changing the rules. What made the 2-dimensional simplicial complex in Figure 7 problematic is that in $\text{Bd}\, K$, the vertex v is not contained in precisely two edges. If the vertex v were contained in one edge, then $\text{Bd}\, K$ would be a 1-dimensional pseudomanifold with boundary, and the problem would not be repairable. However, given that v is contained in four edges, we can get out of harm's way by noting that four, like two but unlike one, is an even number. So, we can try to modify the definition of 2-dimensional pseudomanifold by looking not at the difference between two and one, but between even and odd.

We might want to say that a 2-dimensional simplicial complex K is a "2-dimensional mod 2 pseudomanifold" if every edge of K is contained in an even number of triangles. It would be tempting to say further that a 2-dimensional simplicial complex K is a "2-dimensional mod 2 pseudomanifold with boundary" if every edge of K is contained in an even number or an odd number of triangles, and that the boundary of K would consist of all edges that are contained in an odd number of triangles each. Of course, the term "2-dimensional mod 2 pseudomanifold with boundary" is very silly, because every edge in every 2-dimensional simplicial complex is contained in some finite number of edges (recall that we are restricting attention to finite simplicial complexes), and any whole number is either even or odd. However, the boundary aspect of this proposed term makes sense nonetheless, and so we are led to the following definition.

Definition. Let K be a finite 2-dimensional simplicial complex. The **boundary** of K, denoted $\mathrm{Bd}\, K$, is the collection of all edges of K that are contained in an odd number of triangles each. △

Observe that $\mathrm{Bd}\, K$ is a (possibly empty) 1-dimensional subcomplex of K. What we thought of calling a "2-dimensional mod 2 pseudomanifold" above is simply a simplicial complex with empty boundary, and therefore we do not need this proposed term.

This definition of the boundary of an arbitrary 2-dimensional simplicial complex (and the analogous definition in higher dimensions) is known in the literature (see for example [**13**, p. 136]), though the author came upon this definition independently for precisely the considerations stated above. (However, as previously mentioned, although this notion of boundary of finite simplicial complexes appears in various places in the literature, the only place where the author found this definition used in formulating an exact analog of the NRT was in [**12**].)

We now turn to the topological version of the above approach to boundary, starting with the following notation.

Definition. For each nonnegative integer i, let T_i denote the space obtained by gluing together i copies of the half-open interval $[0, 1)$ at the point $\{0\}$ in each. We take T_0 to be a single point. Let $*$ denote the point of T_i where the half-open intervals meet. △

In Figure 8 we see T_i for $i \in \{0, 1, 2, 3\}$, and in Figure 9 we see part of $T_3 \times \mathbb{R}$, which we will need in the following definition.

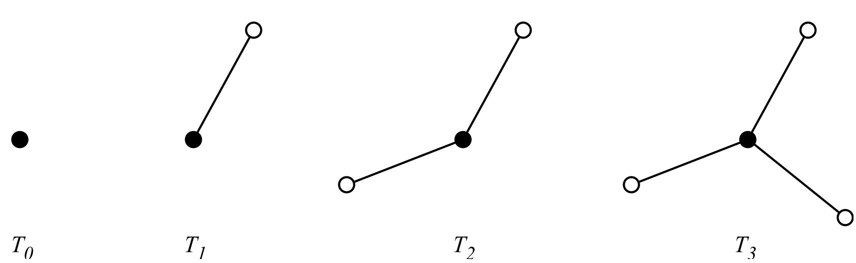

Figure 8.

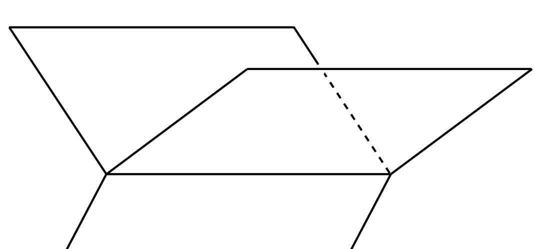

Figure 9.

Definition. Let X be a topological space that is homeomorphic to the underlying space of a 2-dimensional simplicial complex. For each nonnegative integer r, we define the subset $C_r(X)$ of X to be

$$C_r(X) = \{x \in X \mid x \text{ has a neighborhood homeomorphic to } T_r \times \mathbb{R}, \\ \text{where the homeomorphism takes } x \text{ into } \{*\} \times \mathbb{R}\}.$$

The **boundary** of X, denoted ∂X, is defined by

$$\partial X = \operatorname{cl} \bigcup_{r \text{ is odd}} C_r(X),$$

where cl denotes closure. △

If we let X be the object seen in Figure 4, then ∂X is the circle along which the three original disks were glued (the boundaries of the four small disks that were removed are not part of ∂X, because of the tubes). On the other hand, if Y is the analogous construction but starting with four disks that are glued along a circle, then $\partial Y = \emptyset$.

We offer the following comments about the above definition, omitting the proofs.

Remark 2.1. (1) The sets $C_r(X)$ are topological invariants of X, and are disjoint; this fact can be seen by using local homology around each point in X.

(2) For each $r \neq 2$, the set $C_r(X)$ is a finite disjoint union of arcs (without endpoints) and simple closed curves.

(3) The set ∂X is a topological invariant of X.

(4) If M is a surface with boundary, then ∂M is just the standard boundary of M as a surface with boundary.

(5) If K is a 2-dimensional simplicial complex, then $\partial |K| = |\operatorname{Bd} K|$. ◊

3. THE GENERALIZED NO RETRACTION THEOREM.

Having defined the relevant notion of boundary, we are now ready to state and prove the generalized no retraction theorem.

Theorem 3.1. *Let X be a topological space that is homeomorphic to the underlying space of a finite 2-dimensional simplicial complex. Then there is no retraction $r: X \to \partial X$.*

Proof. Suppose that X is homeomorphic to $|K|$, for some finite 2-dimensional simplicial complex K. By Remark 2.1 (5), it will suffice to show that there is no retraction $r: |K| \to |\operatorname{Bd} K|$.

If $\operatorname{Bd} K = \emptyset$, then there is no map of any sort $|K| \to |\operatorname{Bd} K|$, so assume $\operatorname{Bd} K \neq \emptyset$. Suppose to the contrary that there is a retraction $r: |K| \to |\operatorname{Bd} K|$. We will obtain a contradiction.

Step 1: Let Δ be an equilateral triangle with sides of length 1, such that Δ is disjoint from K. Let η be an edge of Δ, and let w be the vertex of Δ not in η. Let σ be an edge in $\operatorname{Bd} K$ (which exists because we are assuming that $\operatorname{Bd} K \neq \emptyset$). Define a map $h: |K| \to \Delta$ by mapping σ affine linearly onto η, by mapping every vertex in $|K| - \sigma$ to w, and extending affine linearly over the simplices of K. Note that h is a simplicial map, that it is continuous, and that all the edges of K (including all of $\operatorname{Bd} K$) are mapped into $\partial\Delta$. It is seen that the only edge of $\operatorname{Bd} K$ mapped by h onto η is σ. Let $r' = h \circ r$. Because r maps $|K|$ into $|\operatorname{Bd} K|$, then $r'(|K|) \subseteq |\partial\Delta|$. Because $r|_{|\operatorname{Bd} K|}$ is the identity map, it follows that $(r'|_{|\operatorname{Bd} K|})^{-1}(\eta) = \sigma$.

We know that r' is continuous, and hence the compactness of $|K|$, together with a standard theorem in real analysis (see for example [**9**, p. 176]), implies that r' is uniformly continuous. Hence there is some number $\delta > 0$ such that if $z, w \in |K|$ are any two points such that $\|z - w\| < \delta$, then $\|r'(z) - r'(w)\| < 1/8$. Let L be a subdivision of K such that the diameter of any triangle of L is less than δ. (It is intuitively evident that such a subdivision can be found, and it can be proved rigorously by using the mth barycentric subdivision for large enough m.) Using Remark 2.1 (5), it is seen that $|\operatorname{Bd} L| = |\operatorname{Bd} K|$.

Define a map $f\colon |L| \to \Delta$ by letting $f(v) = r'(v)$ for all vertices v of L, and extending f affine linearly over the simplices of L. Because r is the identity on $|\operatorname{Bd} K|$, and because $r'(|K|) \subseteq |\partial\Delta|$, it follows that f maps all vertices of L, and all edges of $\operatorname{Bd} L$, into $|\partial\Delta|$. It is not necessarily the case that f maps all edges and triangles of L into $|\partial\Delta|$. However, using the definition of L, and the fact that f is an affine linear map on each simplex of L, it follows that if τ is a triangle in L, then the diameter of $f(\tau)$ is less than $1/8$. Hence, because all vertices of L are mapped into $|\partial\Delta|$, every triangle of L that is not mapped into an edge of $|\partial\Delta|$ must have its image located in one of the three equilateral triangles with altitudes of length $1/8$ located at the vertices of Δ. That is, the image of L is contained in the edges of Δ together with the three shaded triangles seen in Figure 10.

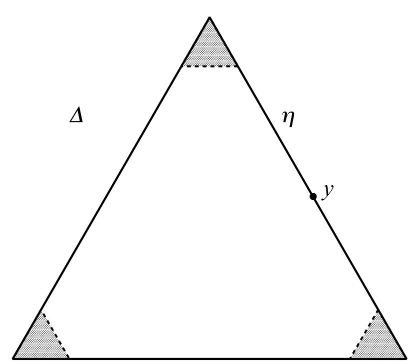

Figure 10.

Choose a point y in the relative interior of η such that it is not the image of any vertex of L under f, and such that it is within $1/8$ of the midpoint of η; given that L has finitely many vertices, such a point y can always be found.

Because $(r'|_{|\operatorname{Bd} K|})^{-1}(\eta) = \sigma$, and because r' maps σ homeomorphically onto η, it follows that $(f|_{|\operatorname{Bd} L|})^{-1}(\{y\}) = (f|_{|\operatorname{Bd} K|})^{-1}(\{y\})$ has one point in it.

Step 2: The choice of y, together with fact that f maps each simplex of L affine linearly, implies that $f^{-1}(\{y\})$ can be thought of as a graph G, where the vertices of G consist of all nonempty intersections of $f^{-1}(\{y\})$ with the relative interiors of edges of L, and the edges of G consist of all the nonempty intersections of $f^{-1}(\{y\})$ with the triangles of L. Those triangles of L that contain an edge of G are precisely those mapped by f into η and have y in the relative interior of their images; no such triangle contains more than one edge.

Let v be a vertex of G. We know that $v \in f^{-1}(\{y\})$, and that v is in the relative interior of an edge μ of L. Let τ be a triangle of L that contains μ. Given the distance of y from the three shaded triangles in Figure 10, together with the fact that the diameter of $f(\tau)$ is less than $1/8$, it follows that $f(\tau)$ is contained in η. Hence τ contains an edge of G. Therefore the number of edges of G that contain v equals the number of triangles of L containing μ. It follows from the definition of $\operatorname{Bd} L$ that v is contained in an odd number of edges of G if and only if μ is in $\operatorname{Bd} L$.

It follows that the number of vertices of G contained in odd numbers of edges of G is equal to the number of points in $(f|_{|\operatorname{Bd} L|})^{-1}(\{y\})$. Because G has straight line edges, it therefore has no loops, and hence we can apply Lemma 1.3 to deduce that the number of points in $(f|_{|\operatorname{Bd} L|})^{-1}(\{y\})$ is an even number. We have thus reached a contradiction, because in Step 1 we saw that $(f|_{|\operatorname{Bd} L|})^{-1}(\{y\})$ had one point in it. ■

Using the fact that compact surfaces with boundary can be triangulated, combined with Remark 2.1 (4), we immediately deduce the following.

Corollary 3.2. *Let X be a compact surface with boundary. Then there is no retraction $r\colon X \to \partial X$.*

ACKNOWLEDGMENTS. I would like to thank David Henderson for introducing me to some ideas that found their way into this proof many years later; the referees, for some very helpful suggestions; and the Einstein Institute of Mathematics at the Hebrew University of Jerusalem, for their hospitality when parts of this paper were written.

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