COMBINATORIAL RICCI CURVATURE FOR POLYHEDRAL SURFACES AND POSETS

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Abstract. The combinatorial Ricci curvature of Forman, which is defined at the edges of a CW complex, and which makes use of only the face relations of the cells in the complex, does not satisfy an analog of the Gauss-Bonnet Theorem, and does not behave analogously to smooth surfaces with respect to negative curvature. We extend this curvature to vertices and faces in such a way that the problems with combinatorial Ricci curvature are mostly resolved. The discussion is stated in terms of ranked posets.

1. Introduction

There have been a number of discrete analogs of the curvature of smooth surfaces and manifolds. The oldest such analog is the angle defect (also known as the angle deficiency) at a vertex $v$ of a triangulated polyhedral surface $M$, which is given by $d(v, M) = 2\pi - \sum_{\alpha \ni v} \alpha$, where the $\alpha$ are the angles at $v$ of the triangles containing $v$. This curvature function goes back at least as far as Descartes (see [8]). The angle defect satisfies various properties one would expect a curvature function to satisfy, notably an analog of the Gauss-Bonnet Theorem, which is that $\sum_{v \in M} d(v, M) = 2\pi \chi(M)$, where the summation is over all the vertices of $M$, and where $\chi(M)$ is the Euler characteristic of $M$.

The angle defect, and related constructs involving sums of angles in polyhedra, have been widely studied in dimension 2 and higher, both for convex polytopes, for example in [22] and [13], and more generally, for example in [2], [7], [14] and [4].

The above discrete analogs of smooth curvature are geometric in nature, making use of interior or exterior angles in simplices and polyhedral cells. It would be interesting to know if there is a purely combinatorial definition of curvature, making use of only the face relations of the cells in the complex, that satisfies properties analogous to smooth curvature. We consider here...
the combinatorial analog of curvature, called combinatorial Ricci curvature, which is defined in [9]. (There are other definitions of discrete curvature that make use of the term “Ricci” in their names, such as the “simplicial Ricci tensor” of [1] and the “discrete Ricci curvature” of [10], but these approaches are very different from [9]). An earlier approach that is more comparable to combinatorial Ricci curvature is the combinatorial analog of curvature given in [24] and [25], and we discuss the latter briefly in Section 4; this same approach, though with a factor of \( \frac{1}{2} \), was used many years later in the context of graphs by [15], and by a number of papers that refer to that one, perhaps unaware that the formula had appeared in [24].

Some, though unfortunately not all, of the properties of smooth curvature have analogs for combinatorial Ricci curvature. An analog of Myers’ Theorem for combinatorial Ricci curvature is given in Theorem 6.1 of [9], which says that if \( K \) is an appropriately nicely behaved CW complex, and if the combinatorial Ricci curvature is positive at every edge in \( K \), then \( \pi_1(K) \) is finite; see [20] for the original smooth version of this theorem. Theorem 7.2 of [9] shows that for any simplicial complex \( K \) of dimension at least 2 that is a combinatorial manifold, there is a subdivision \( M \) of \( K \) such that \( \text{Ric}(e) < 0 \) for every edge \( e \) of \( M \); see [11], [12], [6], [16] and [17] for the smooth analog of that result for Ricci curvature of Riemannian manifolds in dimensions 3 or higher.

The analogy between combinatorial Ricci curvature and smooth curvature breaks down in dimension 2, because there can be no smooth analog of Theorem 7.2 of [9] in dimension 2, due to the Gauss-Bonnet Theorem for smooth surfaces, which implies that a smooth surface with non-negative Euler characteristic cannot have curvature that is everywhere negative. Combinatorial Ricci curvature, therefore, does not satisfy an analog of the Gauss-Bonnet Theorem, and that again shows that combinatorial Ricci curvature is not entirely analogous to smooth curvature.

The purpose of this note is to offer a way to resolve this anomaly of combinatorial Ricci curvature in dimension 2. Specifically, we describe a way to extend combinatorial Ricci curvature to all cells of a 2-dimensional polyhedral complex, and we prove an analog of the Gauss-Bonnet Theorem for this extended curvature. We also provide an answer to the question of everywhere negative curvature in the orientable case.

Combinatorial Ricci curvature is the 1-dimensional case of a more general definition of curvature in [9] that applies to cells in all dimensions, and which is defined as follows. Let \( \alpha \) and \( \eta \) be \( p \)-cells in a CW complex. The \( p \)-cells \( \alpha \) and \( \eta \) are called parallel neighbors if they are either both the faces of a common \((p+1)\)-cell, or they both have a common \((p-1)\)-face, but not both. The curvature at \( \alpha \) is then defined, in the notation of [9], by

\[
\#\{(p+1)\text{-cells } \beta > \alpha\} + \#\{(p-1)\text{-cells } \gamma < \alpha\} - \#\{\text{parallel neighbors of } \alpha\},
\]

(1.1)
where $<$ denotes the relation of being a face. (In [9] the above definition is also given with weights on the cells, though we do not do that here.) Combinatorial Ricci curvature is the special case of this curvature when $p = 1$, and is denoted $\operatorname{Ric}(e)$ for every edge $e$ of the CW complex.

We will restrict our attention to 2-dimensional cell complexes. From the point of view of combinatorial Ricci curvature and the fundamental group, restricting to dimension 2 is no loss, because both are computed entirely in the 2-skeleton of a cell complex.

Because Equation (1.1) uses only the face relations of the cells of a CW complex, it is more clear, and slightly more general, to formulate our discussion in the context of posets, which we do in Section 2; in Section 3 we will return to 2-dimensional polyhedral complexes.

Although the definition of curvature in [9] carries over directly to ranked posets, as long as every element covers, and is covered by, finitely many elements, the approach we take here uses a slight variant of that definition, where we replace the number of parallel neighbors with the more convenient set of all neighbors (which means that non-parallel neighbors are double counted).

2. **Discrete Curvature on Ranked Posets**

We assume that the reader is familiar with basic properties of posets. See [23, Chapter 3] for details. Let $P$ be a poset. We let $<$ denote the partial order relation on $P$, and we write $a < b$ if $b$ covers $a$. A function $\rho: P \to \{0, 1, \ldots, r\}$ for some $r \in \mathbb{N} \cup \{0\}$ is a rank function for $P$ if it satisfies the following conditions: for $a, b \in P$, if $a$ is a minimal element then $\rho(a) = 0$, and if $a < b$ then $\rho(a) + 1 = \rho(b)$. A poset is ranked if it has a rank function. If a poset has a rank function, the rank function is unique. The rank of such a poset is the smallest possible $r$. If $P$ is ranked and has rank $r$, and if $i \in \{0, 1, \ldots, r\}$, let $P_i = \{x \in P \mid \rho(x) = i\}$ and $F_i = |P_i|$.

We note that a ranked poset need not be graded, using the terminology of [23, Chapter 3]; the term “ranked” is used in [21, Section 11.1.3], though we use the definition of a rank function given in [23, Chapter 3].

Let $P$ be a poset. The order complex of $P$, denoted $\Delta(P)$, is the simplicial complex with a vertex for each element of $P$, and a simplex for each non-empty chain of elements of $P$. It is a standard fact that this construction yields a simplicial complex. Suppose that $P$ is finite. The Euler characteristic of $P$, denoted $\chi(P)$ is defined by $\chi(P) = \chi(\Delta(P))$.

The reason to define $\chi(P)$ as $\chi(\Delta(P))$ is that a finite poset in general has no natural rank function, in contrast to the simplicial complex $\Delta(P)$, which is naturally ranked by the dimensions of the simplices, and this natural rank function is what allows the Euler characteristic of simplicial complexes to be defined.
Suppose, however, that \( P \) is a finite ranked poset. Then there is a more direct approach to defining the Euler characteristic of \( P \), as given in the following definition.

**Definition 2.1.** Let \( P \) be a finite ranked poset of rank \( r \). The **ranked Euler characteristic** of \( P \) is the number

\[
\chi_{g}(P) = \sum_{i=0}^{r} (-1)^{i} F_{i}.
\]

If \( P \) is the face poset of a finite regular CW complex, or in particular a polyhedral complex, or simplicial complex, then \( \chi_{g}(P) = \chi(P) \). In general, however, it is not the case that \( \chi_{g}(P) \) and \( \chi(P) \) are equal.

We also need the following definition regarding posets.

**Definition 2.2.** Let \( P \) be a poset. The poset \( P \) is **covering-finite** if for any \( a \in P \), there are finitely many \( b \in P \) such that \( a \prec b \), and there are finitely many \( c \in P \) such that \( c \prec a \).

Throughout this section, let \( P \) be a covering-finite ranked poset of rank \( r \). Let \( i \in \{0, 1, \ldots, r\} \), and \( x \in P_{i} \). Let

\[
A_{i}(x) = |\{y \in P_{i+1} \mid x \prec y\}|, \quad B_{i}(x) = |\{z \in P_{i-1} \mid z \prec x\}|
\]

\[
U_{i}(x) = \sum_{y \succ x} B_{i+1}(y), \quad D_{i}(x) = \sum_{z \prec x} A_{i-1}(z),
\]

and

\[
N_{i}(x) = |\{w \in P_{i} \mid x \prec v \text{ and } w \prec v \text{ for some } v \in P_{i+1}\}|
\]

\[
\triangle \{w \in P_{i} \mid u \prec x \text{ and } u \prec w \text{ for some } u \in P_{i-1}\}
\]

where \( \triangle \) denotes symmetric difference, and summation over the empty set is taken to be zero.

The reader can verify the following equalities:

\[
\sum_{x \in P_{i}} A_{i}(x) = \sum_{y \in P_{i+1}} B_{i+1}(y) \quad (2.1)
\]

\[
\sum_{x \in P_{i}} U_{i}(x) = \sum_{y \in P_{i+1}} [B_{i+1}(y)]^{2} \quad (2.2)
\]

\[
\sum_{x \in P_{i}} D_{i}(x) = \sum_{z \in P_{i-1}} [A_{i-1}(z)]^{2}. \quad (2.3)
\]

Equation (1.1) in the case \( p = 1 \), which defines combinatorial Ricci curvature, can be rewritten as

\[
\text{Ric}(e) = A_{1}(e) + B_{1}(e) - N_{1}(e) \quad (2.4)
\]

for all \( e \in P_{1} \).

We now define our discrete curvature functions on covering-finite ranked posets of rank 2.
Definition 2.3. Let $P$ be a covering-finite ranked poset of rank 2. For each $i \in \{0, 1, 2\}$, let $R_i : P_i \to \mathbb{R}$ be defined by

$$R_0(v) = 1 + \frac{3}{2}A_0(v) - [A_0(v)]^2,$$

$$R_1(e) = 1 + 6A_1(e) + \frac{3}{2}B_1(e) - U_1(e) - D_1(e),$$

$$R_2(\sigma) = 1 + 6B_2(\sigma) - [B_2(\sigma)]^2,$$

for all $v \in P_0$ and $e \in P_1$ and $\sigma \in P_2$. \hfill \triangle$

We will see in Lemma 3.2 that for certain posets, including the face posets of all 2-dimensional polyhedral complexes, the function $R_1$ equals Ric.

The choice of coefficients in Definition 2.3, particularly 6 and $\frac{3}{2}$, may seem unmotivated. They were chosen simply because they relate properly to combinatorial Ricci curvature. A slight variation in the coefficients can also be used, but the above choice appears to be the simplest possible.

The following analog of the Gauss-Bonnet Theorem is a trivial consequence of Definition 2.3 and Equations (2.1)–(2.3); the details are left to the reader.

Theorem 2.4. Let $P$ be a finite ranked poset of rank 2. Then

$$\sum_{v \in P_0} R_0(v) - \sum_{e \in P_1} R_1(e) + \sum_{\sigma \in P_2} R_2(\sigma) = \chi_g(P).$$

We now turn to a less trivial result, which says something about the nature of the poset $P$ in the case that the $R_1$ is everywhere positive, somewhat analogously to Theorem 6.1 of [9]. Our result is weaker than that theorem, due to the fact that not all posets are as nicely behaved as the face posets of CW complexes.

In the case of Gaussian curvature of compact smooth surfaces, the classical Gauss-Bonnet Theorem implies that if the curvature is everywhere positive, then the Euler characteristic of the surface is positive. A similar result holds for the polyhedral curvature defined in [2], and for the combinatorial approach of [24] and [25] (see Section 4 for a brief discussion of that approach). Unfortunately, because of the negative coefficient for the $R_1$ terms in Theorem 2.4, it is not possible to deduce from this version of the Gauss-Bonnet Theorem that if each of $R_0$, $R_1$ and $R_2$ are everywhere positive, then the Euler characteristic is positive. It turns out, as we now see, that it is nonetheless true that positive $R_1$ implies that the ranked Euler characteristic is positive. In fact, all that is required is that the average value of $R_1$ is positive (in contrast to Theorem 6.1 of [9], which requires Ric to be positive everywhere).

We start with a definition.

Definition 2.5. Let $P$ be a finite ranked poset of rank 2. Let

$$\bar{R}_1 = \frac{1}{F_1} \sum_{e \in P_1} R_1(e), \quad \bar{A}_1 = \frac{1}{F_1} \sum_{e \in P_1} A_1(e), \quad \bar{B}_1 = \frac{1}{F_1} \sum_{e \in P_1} B_1(e).$$
The poset \( P \) is sufficiently covered if

\[
[\bar{A}_1 + \bar{B}_1]^2 - 6\bar{A}_1 - \frac{3}{2}\bar{B}_1 - 1 \geq 0.
\]

**Remark 2.6.** It is straightforward to verify that if \( P \) is a finite ranked poset of rank 2 and if \( \bar{B}_1 = 2 \), then \( P \) is sufficiently covered if and only if \( \bar{A}_1 \geq 2 \); in particular, that would hold for the face poset of any simplicial surface, and more generally for any polyhedral map of a surface (as defined in Section 21.1 of [5]). If \( \bar{B}_1 \geq \frac{20}{9} \), then \( P \) is sufficiently covered regardless of the value of \( \bar{A}_1 \).

\( \triangleq \)

In the following theorem, our analog of Theorem 6.1 of [9], we restrict our attention to sufficiently covered posets. After the theorem, we will give a simple example that shows the necessity of some such restriction. A relation between similar average values and the topology of 3-manifolds is discussed in [18], so it is not surprising that such averages are used here as well; it is not clear whether the results of [18] are related to our approach. In [18] the averages take place in simplicial complexes, so \( \bar{B}_1 = 2 \), and therefore only \( \bar{A}_1 \) is considered. Moreover, while there is a very simple formula for \( \bar{A}_1 \) in a simplicial complex, as used in [18], that is not the case for posets that are not the face posets of simplicial complexes.

**Theorem 2.7.** Let \( P \) be a finite sufficiently covered ranked poset of rank 2. If \( \bar{R}_1 > 0 \), then \( \chi_g(P) > 0 \).

**Proof.** Suppose that \( \bar{R}_1 > 0 \). By Equations (2.2) and (2.3), we see that

\[
\frac{1}{2} \bar{R}_1 F_1 < \bar{R}_1 F_1 = \sum_{e \in P_1} R_1(e) = \sum_{e \in P_1} \left[ 1 + 6A_1(e) + \frac{3}{2}B_1(e) - U_1(e) - D_1(e) \right] = F_1 + 6F_1 \bar{A}_1 + \frac{3}{2}F_1 \bar{B}_1 - \sum_{\sigma \in P_2} [B_2(\sigma)]^2 - \sum_{v \in P_0} [A_0(v)]^2,
\]

and hence

\[
\sum_{v \in P_0} [A_0(v)]^2 + \sum_{\sigma \in P_2} [B_2(\sigma)]^2 < F_1 \left[ 1 - \frac{1}{2} \bar{R}_1 + 6\bar{A}_1 + \frac{3}{2}\bar{B}_1 \right].
\]

Let \( d = 1 - \frac{1}{2} \bar{R}_1 + 6\bar{A}_1 + \frac{3}{2}\bar{B}_1 \). The inequality

\[
\frac{1}{n} \left[ \sum_{i=1}^{n} a_i \right]^2 \leq \sum_{i=1}^{n} (a_i)^2
\]

implies that

\[
\frac{1}{F_0} \left[ \sum_{v \in P_0} A_0(v) \right]^2 + \frac{1}{F_2} \left[ \sum_{\sigma \in P_2} B_2(\sigma) \right]^2 < F_1 d.
\]
By Equation (2.1) we deduce that
\[
\frac{1}{F_0} \left[ \sum_{e \in P_1} B_1(e) \right]^2 + \frac{1}{F_2} \left[ \sum_{e \in P_1} A_1(e) \right]^2 < F_1 d.
\]
Hence
\[
\frac{1}{F_0} [F_1 \bar{B}_1]^2 + \frac{1}{F_2} [F_1 \bar{A}_1]^2 < F_1 d,
\]
and therefore
\[
[\bar{B}_1]^2 \frac{F_1}{F_0} + [\bar{A}_1]^2 \frac{F_1}{F_2} < d.
\]
Let \( a = [\bar{A}_1]^2 \) and \( b = [\bar{B}_1]^2 \). Then
\[
b \frac{F_1}{F_0} + a \frac{F_1}{F_2} < d. \tag{2.5}
\]
Because \( P \) is a ranked poset of rank 2, it follows that \( \bar{A}_1 \neq 0 \) and \( \bar{B}_1 \neq 0 \).
Hence \( a, b > 0 \). Because \( b \frac{F_1}{F_0} + a \frac{F_1}{F_2} > 0 \), then \( d > 0 \).

By hypothesis we know that \( P \) is sufficiently covered and \( \frac{1}{2} R_t > 0 \), and hence
\[
[\bar{B}_1 + \bar{A}_1]^2 \geq 1 + 6 \bar{A}_1 + \frac{3}{2} \bar{B}_1 > 1 - \frac{1}{2} R_1 + 6 \bar{A}_1 + \frac{3}{2} \bar{B}_1,
\]
which is the same as \((\sqrt{a} + \sqrt{b})^2 > d\).

Let
\[
U = \{(x, y) \in \mathbb{R}^2 \mid x > 0 \text{ and } y > 0 \text{ and } ax + by < d\},
\]
and let \( f : \mathbb{R}^2 \to \mathbb{R} \) be defined by \( f(x, y) = (x - 1)(y - 1) \) for all \((x, y) \in \mathbb{R}^2\).
The level curve for \( f \) with value \( c = 1 \) is the hyperbola \( y = \frac{1}{x - 1} + 1 \), and the function \( f \) has value less than 1 between the branches of this hyperbola.
Using the condition \((\sqrt{a} + \sqrt{b})^2 > d\), the reader can verify that the set \( U \) is between the two branches of the hyperbola (find the point on the upper branch of the hyperbola at which the tangent line is parallel to the line \( ax + by = d \)). It follows that \( f(x, y) < 1 \) for all \((x, y) \in U\).

Combining the fact that \( \frac{F_1}{F_0} > 0 \) and \( \frac{F_1}{F_2} > 0 \) with Equation (2.5), we know that \( (\frac{F_1}{F_0}, \frac{F_1}{F_2}) \in U, \) and therefore \( (\frac{F_1}{F_0} - 1)(\frac{F_1}{F_2} - 1) < 1 \). On the other hand, we see that
\[
\left( \frac{F_1}{F_0} - 1 \right) \left( \frac{F_1}{F_2} - 1 \right) = \frac{F_1 - F_0}{F_0} \cdot \frac{F_1 - F_2}{F_2} = \frac{F_2 - \chi_g(P)}{F_0} \cdot \frac{F_0 - \chi_g(P)}{F_2} = \frac{1}{F_0F_2} [\chi_g(P)]^2 - \frac{F_0 + F_2}{F_0F_2} \chi_g(P) + 1.
\]
It now follows that
\[
\frac{1}{F_0F_2} [\chi_g(P)]^2 - \frac{F_0 + F_2}{F_0F_2} \chi_g(P) + 1 < 1,
\]
and hence
\[ [\chi_g(P)]^2 - (F_0 + F_2)\chi_g(P) < 0. \]
Because \( x^2 - (F_0 + F_2)x < 0 \) if and only if \( 0 < x < F_0 + F_2 \), we conclude that \( \chi_g(P) > 0 \). \qed

The hypothesis in Theorem 2.7 that \( P \) is sufficiently covered cannot be dropped. Let \( P \) be the poset shown in Figure 1. Then \( R_1(e) = \frac{5}{2} \), and so \( \bar{R}_1 = \frac{5}{2} \), and yet \( \chi_g(P) = -1 \). Hence, some hypothesis on \( P \) is required for the theorem to hold.

\[ \begin{array}{c}
•
\end{array} \]

\textbf{Figure 1}

The analog of Myers’ Theorem in Theorem 6.1 of \cite{9} has a much stronger conclusion than our Theorem 2.7. Specifically, the main part of the proof of Theorem 6.1 of \cite{9} consists of proving that if an appropriately nicely behaved connected CW complex has everywhere positive combinatorial Ricci curvature that is bounded away from zero, then the CW complex is bounded, in the sense that there is an upper bound on the lengths of paths between vertices. If the CW complex is regular and covering-finite, that would imply that the CW complex is finite. Unfortunately, the analogous result does not hold when \( R_1 \) is everywhere positive and bounded away from zero on a covering-finite ranked poset of rank 2, or even when all three of \( R_0, R_1 \) and \( R_2 \) are everywhere positive and bounded away from zero, as seen in the following example. Let \( Q \) be the poset shown in Figure 2, where the pattern repeats infinitely. Let \( v, e, x \in P \) be the elements shown in the figure. Then \( R_0(v) = \frac{3}{2} \), and \( R_1(e) = 4 \) and \( R_2(x) = 9 \), and yet \( Q \) is infinite. We note that Ric works no better for this poset, because \( \text{Ric}(e) = 2 \).

\[ \begin{array}{c}
•••
\end{array} \]

\textbf{Figure 2}
Finally, we note that whereas in the special case of compact smooth surfaces, the Gauss-Bonnet Theorem immediately implies that if the average curvature is positive then so is the Euler characteristic, it is not so simple in our present context of finite ranked posets of rank 2, because our analog of the Gauss-Bonnet Theorem, Theorem 2.4, makes use of $R_0$, $R_1$ and $R_2$, whereas Theorem 2.7 uses only $R_1$, and hence the latter theorem does not follow from the former.

3. 2-DIMENSIONAL POLYHedral Complexes

We now relate our curvature to the problem with combinatorial Ricci curvature that was mentioned at the start of Section 1.

First, we note that for an arbitrary covering-finite ranked poset of rank 2, it is not necessarily the case that $R_1$ equals Ric. For example, let $P$ be the poset shown in Figure 1. Then $R_1(e) = \frac{5}{2}$, but $\text{Ric}(e) = 2$.

However, as we now show, it is the case that $R_1$ equals Ric for the following class of posets.

**Definition 3.1.** Let $P$ be a ranked poset of rank 2. The poset $P$ is **almost polyhedral** if it is covering-finite, and if the following conditions hold. Let $w \in P_0$, and $a, b \in P_1$ and $\tau \in P_2$. Suppose $a \neq b$.

1. $B_1(a) = 2$.
2. There is at most one $v \in P_0$ such that $v \prec a, b$.
3. There is at most one $\sigma \in P_2$ such that $a, b \prec \sigma$.
4. If $w < \tau$, then $[w, \tau]$ has cardinality four.

The face poset of every 2-dimensional polyhedral complex, and in particular every simplicial complex, is an almost polyhedral poset. The face poset of a polyhedral map on a compact surface is also an almost polyhedral poset. However, the set of almost polyhedral posets neither contains, nor is contained in, the set of face posets of all 2-dimensional regular CW complexes. The poset in Figure 3 is almost polyhedral but not the face poset of a regular CW complex, because if it were, then the boundary of $m$ would be disconnected. On the other hand, The face poset of the CW complex with two vertices, two edges and one disk, seen in Figure 4, is not almost polyhedral. We note that Condition (4) in Definition 3.1 is found in a number of places, such as Proposition 2.2 of [3] and Definition 3.3 of [9].
Lemma 3.2. Let $P$ be an almost polyhedral ranked poset of rank 2. Then $R_1(e) = \text{Ric}(e)$ for all $e \in P_1$.

Proof. Let $e \in P_1$. Let

$$U = \{ a \in P_1 \mid e < \tau \text{ and } a < \tau \text{ for some } \tau \in P_2 \}$$

and

$$V = \{ a \in P_1 \mid v < e \text{ and } v < a \text{ for some } v \in P_0 \}.$$

Then $N_1(e) = |U \triangle V|$.

If $e \in P_0$, and $e < \sigma$, let

$$C_\sigma = \{ a \in P_1 \mid a < \sigma \text{ and } a \notin V \}$$

and if $v \in P_0$ and $v < e$, let

$$D_v = \{ a \in P_1 \mid v < a \text{ and } a \notin U \}.$$ 

Then $U - V = \bigcup_{\sigma > e} C_\sigma$ and $V - U = \bigcup_{v < e} D_v$. Observe that $e \in U \cap V$. It follows that $e \notin C_\sigma$ for all $\sigma \in P_2$ such that $e < \sigma$, and $e \notin D_v$ for all $v \in P_0$ such that $v < e$.

Let $u, w \in P_0$. Suppose that $u, w < e$ and $u \neq w$. We claim that $D_u \cap D_w = \emptyset$ and that $|D_u| = A_0(u) - A_1(e) - 1$.

Let $d \in D_u \cap D_w$. Then $u, w < d$, and because $d \neq e$, we have a contradiction to Condition (2) of Definition 3.1. Hence $D_u \cap D_w = \emptyset$.

Let $S = \{ a \in P_1 \mid u < a \}$. Then $S = \{ a \in P_1 \mid u < a \text{ and } a \notin U \} \cup \{ a \in P_1 \mid u < a \text{ and } a \in U - \{ e \} \} \cup \{ e \}$, where this union is disjoint.

Let

$$f: \{ a \in P_1 \mid u < a \text{ and } a \in U - \{ e \} \} \to \{ \sigma \in P_2 \mid e < \sigma \}$$

be defined as follows. Let $m \in \{ a \in P_1 \mid u < a \text{ and } a \in U - \{ e \} \}$. By the definition of $U$ there is some $\tau \in P_2$ such that $m, e < \tau$. Then $\tau$ is unique by Condition (3) of Definition 3.1, and let $f(m) = \tau$. Let $n \in \{ a \in P_1 \mid u < a \text{ and } a \in U - \{ e \} \}$, and suppose that $f(m) = f(n)$. Then $u < m, e, n$ and $m, e, n < f(m) = f(n)$, which contradicts Condition (4) of Definition 3.1, unless $m = n$. Hence $f$ is injective. Let $\eta \in \{ \sigma \in P_2 \mid e < \sigma \}$. Then $u < e < \eta$, and by Condition (4) of Definition 3.1 there is a unique $c \in P_1$.
such that $c \neq e$ and $u \prec c \prec \eta$. Then $c \in \{a \in P_1 \mid u \prec a$ and $a \in U - \{e\}\}$ and $f(c) = \eta$. Hence $f$ is surjective.

Because $f$ is bijective, we see that

$$|S| = |\{a \in P_1 \mid u \prec a$ and $a \notin U\}| + |\{\sigma \in P_2 \mid e \prec \sigma\}| + 1,$$

which implies that $A_0(u) = |D_u| + A_1(e) + 1$, and hence $|D_u| = A_0(u) - A_1(e) - 1$.

Using a similar argument as above, together with the fact that $B_1(e) = 2$, which holds by Condition (1) of Definition 3.1, it is seen that if $\eta, \tau \in P_2$, and if $e \prec \eta, \tau$ and $\eta \neq \tau$, then $C_\eta \cap C_\tau = \emptyset$ and $|C_\eta| = B_2(\eta) - 3$.

It follows that

$$N_1(e) = |U - V| + |V - U| = \sum_{e \prec v} |C_v| + \sum_{v \prec e} |D_v|$$

$$= \sum_{e \prec v} [B_2(v) - 3] + \sum_{v \prec e} [A_0(v) - A_1(e) - 1].$$

Equation (2.4) then yields

$$Ric(e) = A_1(e) + B_1(e) - N_1(e)$$

$$= A_1(e) + 2 - \sum_{e \prec v} [B_2(v) - 3] - \sum_{v \prec e} [A_0(v) - A_1(e) - 1]$$

$$= A_1(e) + 2 - \sum_{e \prec v} B_2(v) + 3A_1(e) - \sum_{v \prec e} A_0(v) + 2A_1(e) + 2$$

$$= 4 + 6A_1(e) - U_1(e) - D_1(e)$$

$$= 1 + \frac{3}{2}B_1(e) + 6A_1(e) - U_1(e) - D_1(e) = R_1(e).$$

Combining Theorem 2.4, Lemma 3.2 and Remark 2.6, we now see that by having $R_0$ and $R_2$ available to us, there is a Gauss-Bonnet Theorem for almost polyhedral ranked posets of rank 2 that incorporates $Ric$.

**Corollary 3.3.** Let $P$ be a finite almost polyhedral ranked poset of rank 2. Then

$$\sum_{v \in P_0} R_0(v) - \sum_{e \in P_1} Ric(e) + \sum_{\sigma \in P_2} R_2(\sigma) = \chi_g(P).$$

Restricting our attention to polyhedral complexes allows us to make use of the standard Euler characteristic. If $K$ is a polyhedral complex, we let $K^{(i)}$ denote the collection of all $i$-cells of $K$, for each $i \in \{0, 1, 2\}$.

**Corollary 3.4.** Let $K$ be a 2-dimensional polyhedral complex. Then

$$\sum_{v \in K^{(0)}} R_0(v) - \sum_{e \in K^{(1)}} Ric(e) + \sum_{\sigma \in K^{(2)}} R_2(\sigma) = \chi(P).$$

The following corollary is deduced immediately from from Theorem 2.7, Lemma 3.2 and Remark 2.6, together with the fact that $B_1 = 2$ in an almost polyhedral ranked poset of rank 2.
Corollary 3.5. Let \( P \) be a finite almost polyhedral ranked poset of rank 2. Suppose that \( \bar{A}_1 \geq 2 \). If the average value of \( \text{Ric} \) is positive, then \( \chi_g(P) > 0 \).

In a polyhedral surface we have \( \bar{A}_1 = 2 \), and hence the following holds.

Corollary 3.6. Let \( K \) be a compact connected polyhedral surface. If the average value of \( \text{Ric} \) is positive, then \( \chi(K) \geq 0 \), and hence \( \pi_1(K) \) is finite.

The observation in Corollary 3.6 that \( \pi_1(K) \) is finite is trivial, because the classification of compact connected surfaces implies that if a compact connected polyhedral surface has positive Euler characteristic, then it is either \( S^2 \) or \( P^2 \), and in both cases it has finite fundamental group. We stated this trivial conclusion in the corollary only for comparison with Theorem 6.1 of [9]. The latter is a much stronger result, because it allows for complexes that are not surfaces, though of course the proof in [9] is much more substantial.

Next, we turn to the question of everywhere negative curvature on polyhedral surfaces. For flexibility, we use polyhedral maps on surfaces, observing that any triangulation of a compact connected surface can be thought of as a polyhedral map. If \( K \) is a polyhedral map on a surface, we let \( K^{(i)} \) denote the collection of all \( i \)-cells of \( K \), for each \( i \in \{0, 1, 2\} \).

Theorem 3.7. Let \( K \) be a polyhedral map on a compact connected surface.

1. \( R_0, \text{Ric} \) and \( R_2 \) have negative values at all cells of \( K \) if and only if \( B_2(\eta) \geq 7 \) for all \( \eta \in K^{(2)} \).
2. If \( \chi(K) \geq 0 \), then not all three of \( R_0, \text{Ric} \) and \( R_2 \) can have negative values at all cells of \( K \).
3. If \( \chi(K) < 0 \) and \( K \) is orientable, there is polyhedral map on the underlying space of \( K \) such that \( R_0, \text{Ric} \) and \( R_2 \) have negative values at all cells of the polyhedral map.

Proof. By Lemma 3.2, we will replace \( \text{Ric} \) with \( R_1 \). We ask when each of \( R_0, \bar{R}_1 \) and \( R_2 \) has negative values.

Let \( v \in K^{(0)} \). Then \( R_0(v) < 0 \) means \( 1 + \frac{3}{2} A_0(v) - [A_0(v)]^2 < 0 \), and it is straightforward to verify that this condition holds if and only if \( A_0(v) \geq 2 \), which is always true for a polyhedral map of a surface. Hence \( R_0(v) < 0 \) is always true.

Let \( e \in K^{(1)} \). Because \( K \) is a polyhedral map of a surface then \( A_1(e) = 2 \) and \( B_1(e) = 2 \). Hence \( \bar{R}_1(e) < 0 \) if and only if \( 16 - U_1(e) - D_1(e) < 0 \), which is equivalent to \( \sum_{\sigma < e} A_0(v) + \sum_{\sigma > e} B_2(\sigma) > 16 \). The edge \( e \) has two vertices, and \( A_0(v) \geq 2 \) for all vertices \( v \in K^{(0)} \), and therefore if \( \sum_{\sigma > e} B_2(\sigma) > 12 \) then \( \bar{R}_1(e) < 0 \); this condition on \( \sum_{\sigma > e} B_2(\sigma) \) is not necessary for obtaining \( \bar{R}_1(e) < 0 \). In particular, because \( e \) is contained in two faces, if \( B_2(\sigma) \geq 7 \) for all \( \sigma \in K^{(2)} \) then \( \bar{R}_1(e) < 0 \).

Let \( \sigma \in K^{(2)} \). Then \( R_2(\sigma) < 0 \) means \( 1 + 6B_2(\sigma) - [B_2(\sigma)]^2 < 0 \), and it is straightforward to verify that this condition holds if and only if \( B_2(\sigma) \geq 7 \).

Putting the above considerations together immediately implies Part (1) of this theorem.
Suppose $\chi(K) \geq 0$. It can be verified that $K$ must have a face with no more than 6 edges; details are left to the reader. Part (2) of this theorem then follows immediately from Part (1).

Now suppose that $K$ is orientable and $\chi(K) < 0$. Then the genus of $K$ is greater than or equal to 2. By Theorem 2(a) of [19] there is a polyhedral surface $L$ with the same genus as $K$, and with each face having 3 edges, and each vertex contained in 7 edges; this surface can be rectilinearly embedded in $\mathbb{R}^3$. (Such a polyhedral surface is called equivelar.) Let $M$ be the dual map of $L$. Then each face of $M$ has 7 edges. It follows immediately from Part (1) of this theorem that $R_0, R_1$ and $R_2$ have negative values at all cells of $M$, which is Part (3) of the theorem. □

The glaring omission in Theorem 3.7 is that Part (3) of the theorem treats only the orientable case. It does not appear to be known whether for every non-orientable compact connected surface $K$ with $\chi(K) < 0$, there is a polyhedral map $M$ of the underlying space of $K$ such that every face of $M$ has at least 7 edges. It would be interesting to know if that holds.

4. Comparison with D. Stone’s Approach

The approach taken in [9] is not the only combinatorial approach to curvature of CW complexes. Another (in fact earlier) approach was taken in [24] and [25], where an analog of Myers’ Theorem was proved in dimension 2, using ideas similar to those in [9]; the results of the latter are stronger than the former. We now offer a brief comparison of our approach in the case of polyhedral surfaces with the approach of [24] and [25].

In [9], the combinatorial Ricci curvature is located at the edges; in our approach the discrete curvature for polyhedral surfaces is located at all cells, though from the point of view of the analog of Myers’ Theorem, the curvature of interest is located at the edges. By contrast, the curvature defined in [24] and [25] is located at the vertices, and is defined as follows. Let $K$ be a polyhedral surface, and let $v \in K^{(0)}$. Using our notation, the curvature at $v$ is given by the formula

$$R^*(v) = 2 - \sum_{\sigma \ni v} \left( 1 - \frac{2}{B_2(\sigma)} \right),$$

where the summation is over all 2-cells of $K$ that contain $v$.

It is simple to verify that an analog of the Gauss Bonnet Theorem, which is $\sum_{v \in K^{(0)}} R^*(v) = 2\chi(K)$, holds for this definition of curvature for all polyhedral surfaces. However, in contrast to the analog of the Gauss-Bonnet Theorem in Theorem 2.4, which holds for all finite ranked posets of rank 2, the analog of the Gauss Bonnet Theorem for $R^*$ does not hold for all 2-dimensional simplicial complexes, as examples show; we omit the details.

When $K$ is a polyhedral surface, the number of 2-cells that contain $v$ equals the number of edges that contain $v$, which is $A_0(v)$. Hence we can
trivially rewrite Equation 4.1 as

\[ R^*(v) = 2 - A_0(v) + \sum_{\sigma > v} \frac{2}{B_2(\sigma)}. \]  \hspace{1cm} (4.2)

In contrast to the formula for \( R^* \) given in Equation 4.1, which is precisely as given in [24] and [25], the formula given in Equation 4.2 does satisfy the analog of the Gauss Bonnet Theorem for all 2-dimensional simplicial complexes. However, while the formula in Equation 4.2 can, similarly to combinatorial Ricci curvature, be extended unchanged to the context of covering-finite ranked posets, the analog of the Gauss-Bonnet Theorem does not hold for this definition of \( R^* \) for all finite ranked poset of rank 2, as examples show; again, we omit the details.

We now return to the case where \( K \) is a polyhedral surface. The analog of the Gauss-Bonnet Theorem for \( R^* \), which in contrast to Corollary 3.3 does not have \(-1\) coefficients for any terms, implies that if \( \chi(K) \geq 0 \) then it cannot be the case that \( R^*(w) < 0 \) for all \( w \in K^{(0)} \), which is analogous to Theorem 3.7 (2).

Similarly to one direction of Theorem 3.7 (1), if \( B_2(\eta) \geq 7 \) for all \( \eta \in K^{(2)} \), then

\[ R^*(v) \leq 2 - A_0(v) + \sum_{\sigma > v} \frac{2}{t} = 2 - A_0(v) + \frac{2}{t} A_0(v) \leq 2 - \frac{5}{t} \cdot 3 < 0 \]

for all \( v \in K^{(0)} \), because \( A_0(v) \geq 3 \). (We note, however, that whereas this condition is also a necessary condition in Theorem 3.7 (1), it is not necessary for \( R^*(v) \) for all \( v \in K^{(0)} \); for example, by [19, Theorem 2(c)] there is a polyhedral surface \( M \) such that \( \chi(M) = -8 \), that each face has 5 edges, and each vertex is contained in 4 edges; it is seen that \( R^*(v) < 0 \) for all \( v \in M \).)

The proof of Theorem 3.7 (3) also shows that if \( K \) is orientable and \( \chi(K) < 0 \), there is polyhedral surface \( M \) with the same underlying space as \( K \) such that \( R^*(w) < 0 \) for all \( w \in K^{(0)} \).

Finally, we note that \( R^* \) does not work any better than Ric, or \( R_0 \), \( R_1 \) and \( R_2 \) in relation to a possible analog of Myers’ Theorem for posets. Let \( P \) be the poset shown in Figure 2, and \( v \in P \) be the element shown in the figure. The reader can verify that \( R^*(v) = 3 \). It is therefore not the case that \( R^*(w) \) is positive and bounded away from zero for all \( w \in P_0 \) guarantees that the poset is finite.

References


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