POLYHEDRAL REPRESENTATION OF DISCRETE MORSE FUNCTIONS ON REGULAR CW COMPLEXES AND POSETS

PRELIMINARY DRAFT

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ABSTRACT. It is proved that the critical cells of a discrete Morse function in the sense of Forman on a finite regular CW complex can always be detected by a polyhedral Morse function in the sense of Banchoff on an appropriate embedding in Euclidean space of the barycentric subdivision of the complex. The proof is stated in terms of discrete Morse functions on a class of posets that is slightly broader than the class of face posets of finite regular CW complexes.

1. INTRODUCTION AND STATEMENT OF THE RESULT

In its classical form, Morse theory is concerned with smooth functions on smooth manifolds. See the standard reference [Mil63] for the basics of Morse theory. In addition to the traditional smooth approach, there have also been a number of discrete analogs of Morse theory, two of which have been widely cited in the literature: the first, due to Banchoff, is found in [Ban67], [Ban70] and [Ban83]; the second, due to Forman, is more recent, and is found in [For98a], [For98b] and subsequent papers.

Although Banchoff’s and Forman’s approaches are widely cited, there does not appear to be in the literature a thorough discussion of the relation between these two approaches. Such a lack of discussion is perhaps due to the fact that upon first encounter, the two approaches appear to be quite different. Banchoff considers finite polyhedra embedded in Euclidean space, whereas Forman considers CW complexes (not necessarily embedded). A “Morse function” for Banchoff is a projection onto a straight line in Euclidean space, whereas a “Morse function” for Forman, called a “discrete Morse function,” is a map that assigns a number to each cell of a CW complex, subject to certain conditions. Given a projection map, Banchoff defines an index at each vertex of a polyhedron, but at no other cells, whereas given a discrete Morse function, Forman defines an index for each critical cell, which could be of any dimension. In [Ban67] Banchoff does not define the

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concept of critical vertices vs. ordinary vertices, though he does do so for polyhedral surfaces in in [Ban70], and we will use this approach for all dimensions; for Forman, the distinction between critical cells vs. ordinary cells is of great importance. Finally, Banchoff focuses on relating the index at vertices to polyhedral curvature, whereas Forman focuses on using critical points for the purpose of reconstructing the CW complex up to homotopy type by attaching appropriate cells.

In spite of these apparent differences, the purpose of this note is to prove that there is a very concrete relation between the approaches of Banchoff and Forman, as given in the following theorem. This theorem says that information about critical cells in the sense of Forman can be obtained by Banchoff’s method for an appropriate embedding of the barycentric subdivision of the original finite regular CW complex. We note that in Forman’s method a critical $p$-cell always has index $p$, and so the only question to be asked is whether or not a cell is critical, not what its index is.

We assume that the reader is familiar with Banchoff’s approach as in [Ban67], and Forman’s approach as in [For98a], though we will make use of only the first few sections of the latter paper. We need the following clarification of Banchoff’s method, which is taken from [Ban70]. Let $K$ be a simplicial complex in some $\mathbb{R}^m$, and let $\xi \in S^{m-1}$ be a unit vector. In order to define the index at each vertex of $K$, Banchoff assumes that projection onto the line spanned by $\xi$ gives distinct values for any two vertices of $K$ that are joined by an edge; this condition is true for almost all values of $\xi$. With that assumption, Banchoff defines an index denoted $a(v, \xi)$, for each vertex $v$ of $K$. As mentioned above, Banchoff does not define the terms critical vertex vs. ordinary vertex in [Ban67]. However, in [Ban70], which treats only surfaces, he defines a vertex to be critical if and only if $a(v, \xi) \neq 0$, and we will take that definition as the correct one for higher dimensions as well.

**Theorem 1.1.** Let $C$ be a finite regular CW complex, and let $f$ be a discrete Morse function on $C$. For any sufficiently large $m \in \mathbb{N}$, and for any line in $\mathbb{R}^m$, there is a polyhedral embedding of the barycentric subdivision of $C$ in $\mathbb{R}^m$ such that a cell in $C$ is critical in the sense of Forman with respect to $f$ if and only if its barycenter is critical in the sense of Banchoff with respect to projection onto the line.

For the sake of brevity, we will say “discrete-critical” when we mean “critical in the sense of Forman,” and “polyhedral-critical” when we mean “critical in the sense of Banchoff,” and similarly for ordinary cells and vertices.

The reason we took the barycentric subdivision of the CW complex in Theorem 1.1 is for the following reasons. First, whereas Forman’s method determines whether every cell is critical or ordinary, Banchoff’s method assigns such information only to the vertices, and by taking the barycentric subdivision we obtain a single vertex corresponding to each original cell. Second, the barycentric subdivision of a regular CW complex is a simplicial complex, and simplicial complexes are easier to embed in Euclidean space than a more general cell complex. Third, even if the original CW complex were a simplicial complex, we would still need to take its barycentric subdivision prior to embedding the complex in Euclidean space, because of the following simple example. Let $K$ be a triangle together with its faces,
which is a simplicial complex. The function that assigns to each face of the triangle its dimension is a discrete Morse function, and every face is discrete-critical (as mentioned in [For98a, p. 108]). However, for any embedding of the triangle in Euclidean space prior to barycentric subdivision, it is seen that the projection onto any appropriate line in the Euclidean space takes any point in the interior of an edge to a value lower than one of its vertices, and any such point, when taken as a vertex of a subdivision of the edge, is polyhedral-ordinary. Hence, if we want to recover the discrete-critical cells by projection onto a line in Euclidean space, we need the flexibility of first taking the barycentric subdivision prior to embedding.

Although the motivation for this note was as stated above, it turns out that the only property of regular CW complexes that is needed for the proof of Theorem 1.1 is the fact that the set of cells of a regular CW complex form a ranked poset (partially ordered set) in a natural way, and such a poset has various nice properties. It is therefore more clear, and slightly more general, to formulate and prove our theorem in the context of posets.

We assume that the reader is familiar with basic properties of posets. See [Sta97, Chapter 3] for details. All posets are assumed to be finite. Let $P$ be a poset. We let $<$ denote the partial order relation on $P$, and we write $a < b$ if $b$ covers $a$, and $a \preceq b$ if $a < b$ or $a = b$. We let $P_{<a}$ and $P_{\leq a}$ denote the intervals $\{x \in P \mid x < a\}$ and $\{x \in P \mid x \leq a\}$ respectively.

The order complex of $P$, which is denoted $\Delta(P)$, is the simplicial complex with a vertex for each element of $P$, and a simplex for each non-empty chain of elements of $P$; it is a standard fact that such a construction yields a simplicial complex. If $C \subseteq P$ is a chain (always assumed non-empty), we let $l(C)$ denote the length of the chain, which is one less than the number of elements in the chain.

We need a few additional properties of posets. A standard definition for posets is that a poset $P$ is pure if all maximal chains have the same length; in that case it is possible to define a rank function $\rho : P \to \{0,1,\ldots,r\}$ that satisfies the following conditions: for $a,b \in P$, if $a$ is minimal then $\rho(a) = 0$, and if $a < b$ then $\rho(a) + 1 = \rho(b)$. We will use the mod 2 version of this concept, as given in Part 2 of the following definition, where we follow the terminology of [Brü04, p. 6].

**Definition.** Let $P$ be a finite poset.

1. The poset $P$ is **two-wide** if for any $a,b,c \in P$ such that $a < b < c$, there is some $d \in P$ such that $d \neq b$ and $a < d < c$.
2. The poset $P$ is **parity-graded** if all maximal chains have the same length mod 2; in that case there is a **parity rank function** $\mu : P \to \{0,1\}$ that satisfies the following condition: for $a,b \in P$, if $a$ is minimal then $\mu(a) = 0$, and if $a < b$ then $\mu(a) + 1 \equiv \mu(b)$ (mod 2).
3. Suppose that $P$ is parity-graded. The poset $P$ is **almost-cw** if for every $a \in P$ for which $P_{<a} \neq \emptyset$, we have $\chi(\Delta(P_{<a})) = (-1)^{\mu(a)+1} + 1$.

It is evident that if a poset $P$ is pure then it is parity-graded, but the converse is not true.

Let $X$ be a regular CW complex. Then the face poset of $X$, denoted $P(X)$, is the poset that has one element for each cell of $X$, where the order relation is given by
\( \sigma < \tau \) if \( \sigma \) is in the boundary of \( \tau \), for cells \( \sigma \) and \( \tau \) in \( X \). The poset \( P(X) \) is ranked, where the rank of a cell in \( X \) is its dimension. It is a standard fact that \( \Delta(P(X)) \) and \( X \) have homeomorphic underlying spaces. The topological name for \( \Delta(P(X)) \) is the barycentric subdivision of \( X \); if \( X \) is a simplicial complex, then \( \Delta(P(X)) \) is combinatorially the same as the usual barycentric subdivision of \( X \). These facts about \( P(X) \) may be found in standard references about CW complexes such as [LW69] and [CF67], and in combinatorial references such as [Bjo95] and [Sta97]. It can be seen that \( P(X) \) is two-wide, parity-graded and almost-cw. The first fact is noted in follows from [For98a, Theorem 1.2], the second is straightforward, and the third is true because for each \( \sigma \in P(X) \), the interval \( P(X)_{<\sigma} \) is the set of all cells in the boundary of \( \sigma \), which is a sphere, and hence has the appropriate Euler characteristic.

Although the face poset of a regular CW complex is two-wide, parity-graded and almost-cw, not every poset satisfying these three properties is the face poset of a regular CW complex. For example, let \( P \) be the poset shown in Figure 1. The reader may verify that \( P \) the three properties. However, we note that \( P \) is not the face poset of a regular CW complex, because if it were, then the interval \( P_{<13} \) would be the face poset of the boundary of cell 13, and hence \( \Delta(P_{<13}) \) would be a sphere, and yet \( \Delta(P_{<13}) \) is not connected, as the reader may verify. We also note that \( P \) is parity-graded, but is not pure.

\[ \text{Figure 1} \]

Whereas the original formulation of discrete Morse theory in [For98a] is for discrete Morse functions on CW complexes, the same definition of discrete Morse functions can be phrased for posets.

**Definition.** Let \( P \) be a poset, and let \( f: P \to \mathbb{R} \) be a function.

1. The map \( f \) is a discrete Morse function if the following condition holds: for each \( b \in P \), there is at most one \( a \in P \) such that \( a < b \) and \( f(a) \geq f(b) \), and there is at most one \( c \in P \) such that \( b < c \) and \( f(b) \geq f(c) \).

2. Suppose \( f \) is a discrete Morse function. An element \( b \in P \) is discrete-critical with respect to \( f \) if there is no \( a \in P \) such that \( a < b \) and \( f(a) \geq f(b) \), and there is no \( c \in P \) such that \( b < c \) and \( f(b) \geq f(c) \); otherwise \( b \) is discrete-ordinary with respect to \( f \). \( \triangle \)
The following lemma is just a restatement for posets of Lemma 2.5 of [For98a].

**Lemma 1.2.** Let $P$ be a two-wide poset, and let $f : P \to \mathbb{R}$ be a discrete Morse function. If $b \in P$, there cannot be both some $a \in P$ such that $a \prec b$ and $f(a) \geq f(b)$, and some $c \in P$ such that $b \prec c$ and $f(b) \geq f(c)$.

We note that Lemma 1.2 is not true if the assumption that $P$ is two-wide is dropped. For example, let $P = \{0, 1, 2\}$ have the usual total order, and let $f : P \to \mathbb{R}$ be defined by $f(x) = 2 - x$ for $x \in P$. Then $f$ is a discrete Morse function on $P$, but it does not satisfy the conclusion of the lemma.

Our main theorem is as follows.

**Theorem 1.3.** Let $P$ be a two-wide, parity-graded, almost-cw finite poset and let $f$ be a discrete Morse function on $P$. For any sufficiently large $m \in \mathbb{N}$, and for any line in $\mathbb{R}^m$, there is a polyhedral embedding $\phi : \Delta(P) \to \mathbb{R}^m$ such that $w \in P$ is discrete-critical with respect to $f$ if and only if $\phi(w)$ is polyhedral-critical with respect to projection onto the line.

By the properties of the face poset of a CW complex stated above, it follows that Theorem 1.1 is an immediate corollary of Theorem 1.3, and so we will prove only the latter.

2. PROOF OF THE THEOREM

We start with the following three lemmas, the first two of which are very simple, and the third of which is the bulk of our work.

**Lemma 2.1.** Let $V$ be a finite set with $n$ elements, where $n \geq 1$, and let $f : V \to \mathbb{R}$ be a function. There is a map $\psi : V \to \mathbb{R}^n$ such that $\psi(V)$ spans an $(n-1)$-simplex, and for each vertex $v \in V$, the projection of $\psi(v)$ onto the x-axis equals $f(v)$.

**Proof.** The proof is by induction on $n$. If $n = 1$, let $v$ be the single element of $V$, and then define $\psi(v) \in \mathbb{R}$ to be $\psi(v) = f(v)$. Now suppose the result is true for $n - 1$, where $n \geq 2$. Let $w \in V$, and let $V' = V - \{w\}$. Because $V'$ has at least one element, then by the inductive hypothesis there is a map $\phi : V' \to \mathbb{R}^{n-1}$ such that $\phi(V')$ spans an $(n-2)$-simplex, and for each vertex $v \in V'$, the projection of $\phi(v)$ onto the x-axis equals $f(v)$. We can think of $\mathbb{R}^{n-1}$ as sitting in $\mathbb{R}^n$ in the usual way, and hence we can think of $\phi$ as a map $V' \to \mathbb{R}^n$. Let $\psi : V \to \mathbb{R}^n$ be defined by letting $\psi|_{V'} = \phi$, and letting $\psi(w)$ be a point in $\mathbb{R}^n$ with first coordinate equal to $f(w)$, and last coordinate not equal to zero. Because $\psi(w)$ can be joined to $\psi(V')$, we see that $\psi(V)$ spans an $(n-1)$-simplex, and it is evident by definition that for each vertex $v \in V$, the projection of $\psi(v)$ onto the x-axis equals $f(v)$. 

For the next lemma, we need the following notation. Let $P$ be a poset. If $S \subseteq P$, we let $\text{chains}(S)$ denote the set of non-empty chains in $S$. If $b, s, t \in P$, and if $s \lesssim b$ and $t \lesssim b$, we let

\[
ch(b; s) = \{ C \in \text{chains}(P_{\leq b}) \mid s \in C \}
\]

\[
ch(b; s, t) = \{ C \in \text{chains}(P_{\leq b}) \mid s \in C \text{ and } t \in C \}
\]

\[
ch(b; -s, t) = \{ C \in \text{chains}(P_{\leq b}) \mid s \notin C \text{ and } t \in C \}
\]

\[
ch(b; -s, -t) = \{ C \in \text{chains}(P_{\leq b}) \mid s \notin C \text{ and } t \notin C \}
\]
Lemma 2.2. Let $P$ be a parity-graded, almost-cw finite poset. Let $b \in P$.

1. $\sum_{C \in ch(b; b)} (-1)^{l(C)} = (-1)^{\mu(b)}$.
2. If $a \in P$ and $a \prec b$, then $\sum_{C \in ch(b; a, b)} (-1)^{l(C)} = 0$.
3. If $a \in P$ and $a \prec b$, then $\sum_{C \in ch(b; a)} (-1)^{l(C)} = 0$.

Proof: For Part 1, we have two cases. First, suppose that $P_{< b} = \emptyset$. Then $b$ is a minimal element of $P$, and therefore $\mu(b) = 0$. Also, we see that $ch(b; b) = \{ \{ b \} \}$, and therefore $\sum_{C \in ch(b; b)} (-1)^{l(C)} = (-1)^0 = (-1)^{\mu(b)}$. Second, suppose $P_{< b} \neq \emptyset$. There is a bijective map from $ch(b; b) - \{ \{ b \} \}$ to $\text{chains}(P_{< b})$, where the map is obtained by taking each chain in the former set and removing $b$. This map shortens the length of each chain by 1. Using the definition of the order complex together with the definition of almost-cw, we have

$$\sum_{C \in ch(b; b)} (-1)^{l(C)} = \sum_{D \in \text{chains}(P_{< b})} (-1)^{l(D)} + (-1)^{l(\{ b \})}$$

$$= - \sum_{D \in \text{chains}(P_{< b})} (-1)^{l(D)} + (-1)^0 = - \chi(\Delta(P_{< b})) + 1$$

$$= - \left[ (-1)^{\mu(b)} + 1 \right] + 1 = (-1)^{\mu(b)}.$$ 

For Part 2, let $a \in P$, and suppose that $a \prec b$. Then

$$ch(b; -a, b) = ch(b; b) - ch(b; a, b).$$

There is a bijective map from $ch(b; a, b)$ to $ch(a; a)$, where the map is obtained by taking each chain in the former set and removing $b$. This map shortens the length of each chain by 1. We then use Part 1, together with the fact that $\mu(a) + 1 \equiv \mu(b) \mod 2$, to see that

$$\sum_{C \in ch(b; -a, b)} (-1)^{l(C)} = \sum_{C \in ch(b; b)} (-1)^{l(C)} - \sum_{D \in \text{chains}(b; a, b)} (-1)^{l(D)}$$

$$= \sum_{C \in ch(b; b)} (-1)^{l(C)} - \sum_{D \in \text{chains}(a; a)} (-1)^{l(D) + 1}$$

$$= (-1)^{\mu(b)} - \left[ (-1)^{\mu(a)} \right] = 0.$$ 

The proof of Part 3 is similar to the proof of Part 2, and we omit the details. \qed

Lemma 2.3. Let $P$ be a two-wide finite poset, and let $f$ be a discrete Morse function on $P$. Then there is a discrete Morse function $g$ on $P$ that satisfies the following properties. Let $z, x, y, u \in P$. Suppose that $x \prec y$.

1. If $z < x$ and $g(x) < g(y)$, then $g(z) < g(y)$.
2. If $y < u$ and $g(x) < g(y)$, then $g(x) < g(u)$.
3. If $z \neq x$, then $g(z) \neq g(x)$.
4. An element of $P$ is discrete-critical with respect to $f$ if and only it is discrete-critical with respect to $g$. 

Proof: Let \( a \in P \). We say that \( a \) is up-troubled (respectively short-up-troubled) with respect to \( f \) if there are \( x,y \in P \) such that \( a \prec x \prec y \) (respectively \( a \prec x \prec y \)) and that \( f(x) < f(y) \leq f(a) \). We say that \( a \) is down-troubled (respectively short-down-troubled) with respect to \( f \) if there are \( w,c \in P \) such that \( w \prec c \prec a \) (respectively \( w \prec c \prec a \)) and that \( f(a) \leq f(w) < f(c) \).

Step 1: We will modify \( f \) so that it has no short-up-troubled elements, the modified \( f \) is still a discrete Morse function, and the critical elements are unchanged by the modification. We use the standard result that there is a total order on \( P \) that is consistent with the original partial order \( < \) on \( P \). Suppose that such a total order is chosen. We will proceed recursively according to the total order, modifying \( f \) once for each element of \( P \).

Let \( a \in P \) be the least element of \( P \) with respect to the total order. Then \( a \) is minimal with respect to \( < \). If \( a \) is not short-up-troubled, we do not modify \( f \) at this stage. Now suppose that \( a \) is short-up-troubled. Then there are \( x,y \in P \) such that \( a \prec x \prec y \) and \( f(x) < f(y) \leq f(a) \). Observe that \( a \) and \( x \) are both discrete-critical with respect to \( f \). By the definition of discrete Morse functions, we know that \( x \) is unique (though \( y \) need not be unique), and we also know that \( f(a) < f(z) \) for all \( z \in P \) such that \( a \prec z \) and \( z \neq x \). By Lemma 1.2 we know that if \( b \in P \) and \( x \prec b \), then \( f(x) < f(b) \). We then modify \( f \) by decreasing the value of \( f(a) \) so that it is strictly greater than \( f(x) \), and strictly less than \( f(b) \) for all \( b \in P \) such that \( x \prec b \). The modified \( f \) is still a discrete Morse function, no critical elements have changed as a result of this modification, and now \( a \) is not short-up-troubled.

Now suppose that \( f \) has been modified so that the first \( k-1 \) elements of \( P \) in the total order are not short-up-troubled, that \( f \) is still a discrete Morse function, and that the critical elements have not changed. Let \( e \in P \) be the \( k \)-th element of \( P \) in the total order. If \( w \in P \) and \( w \prec e \), then \( w \) is prior to \( e \) in the total order, and hence \( w \) is not short-up-troubled. If \( e \) is not short-up-troubled, we do not modify \( f \) at this stage. Now suppose that \( e \) is short-up-troubled. Then there are \( x,y \in P \) such that \( e \prec x \prec y \) and \( f(x) < f(y) \leq f(e) \). As before, we know that that \( e \) and \( x \) are both discrete-critical, that \( x \) is unique, that \( f(e) < f(z) \) for all \( z \in P \) such that \( e \prec z \) and \( z \neq x \), and that \( f(x) < f(b) \) for all \( b \in P \) such that \( x \prec b \).

Suppose that there is some \( h \in P \) such that \( h \prec e \) and \( f(x) \leq f(h) \). Because \( P \) is two-wide, there is some \( t \in P \) such that \( t \neq e \) and \( h \prec t \prec x \). Because \( f(x) \leq f(e) \), then by the definition of discrete Morse functions we know that \( f(t) < f(x) \). Because \( f(x) \leq f(h) \), we deduce that \( h \) is short-up-troubled, which is a contradiction. Hence \( f(d) < f(x) \) for all \( d \in P \) such that \( d \prec e \).

We now modify \( f \) by decreasing the value of \( f(e) \) so that it is strictly greater than \( f(x) \), and strictly less than \( f(b) \) for all \( b \in P \) such that \( x \prec b \). Because of the previous paragraph, we see that the modified \( f \) is still a discrete Morse function, no critical elements have changed as a result of this modification, and now \( e \) is not short-up-troubled.

By recursion, we can modify \( f \) so that it has no short-up-troubled elements, and has the same critical elements as it had originally.

Step 2: We prove that the modified \( f \) has no up-troubled elements. Suppose to the contrary that there is some \( a \in P \) that is up-troubled. Then there are \( x,y \in P \) such
that \( a < x < y \) and \( f(x) < f(y) \leq f(a) \). Because \( a \) is not short-up-troubled, then \( a \neq x \). By a standard fact about posets there are \( b_1, b_2, \ldots, b_q \in P \), with \( q \geq 1 \), such that \( a < b_1 < b_2 < \cdots < b_q < x < y \). Without loss of generality, we may assume that \( a \) was chosen so that \( q \) is minimal for all possible up-troubled elements. This minimality implies that \( f(b_j) < f(y) \leq f(a) \) for all \( j \in \{1, \ldots, q\} \). It follows in particular that \( f(b_1) < f(a) \). By Lemma 1.2 we see that \( f(b_1) < f(b_2) \), where we replace \( b_2 \) with \( x \) if \( q = 1 \). Because \( a < b_1 < b_2 \) and \( f(b_1) < f(b_2) < f(a) \), we deduce that \( a \) is short-up-troubled, which is a contradiction.

Step 3: We now modify \( f \) so that it has no short-down-troubled elements, and that the critical elements are unchanged by the modification. The modification is the same as in Step 1, except that it is upside down. The only question is whether we can perform this modification in such a way that it does not cause any elements to become short-up-troubled; if we can make sure that no element becomes short-up-troubled, then by Step 2 no element will be up-troubled either.

We proceed recursively, again using the total order on \( P \) given in Step 1, though this time starting from the greatest element with respect to the total order, and proceeding downward. Let \( q \in P \) be the greatest element of \( P \) with respect to the total order. Modify \( f \) analogously to the way we modified \( f \) at the least element \( a \) in Step 1, so that after the modification \( q \) is not short-down-troubled, the modified \( f \) is still a discrete Morse function, and the critical elements of \( f \) have not changed. This modification of \( f \), which is done by possibly increasing the value of \( f(q) \), cannot cause \( q \) or any element that is less than \( q \) with respect to \( < \) to become short-up-troubled, and because \( q \) is a maximal element with respect to \( < \), there is nothing else that could become short-up-troubled as a result of this modification.

Now suppose that \( f \) has been modified so that the last \( k-1 \) elements of \( P \) in the total order are not short-down-troubled, and that the critical elements have not changed, and there are no short-up-troubled elements. Hence by Step 2 there are no up-troubled elements. Let \( u \in P \) be the \( k \)-th from last in the total order. If \( v \in P \) and \( u < v \), then \( v \) is after \( u \) in the total order, and hence by hypothesis \( v \) is not short-down-troubled. If \( u \) is not short-down-troubled, we do not modify \( f \) at this stage. Now suppose that \( u \) is short-down-troubled. Then there are \( w, c \in P \) such that \( w < c < u \) and \( f(u) \leq f(w) < f(c) \). Analogously to Step 1, we know that \( c \) is unique (though \( w \) need not be unique), that \( f(v) < f(u) \) for all \( v \in P \) such that \( v < u \) and \( v \neq c \), that \( f(h) < f(c) \) for all \( h \in P \) such that \( h < c \), and that \( f(c) < f(p) \) for all \( p \in P \) such that \( u < p \).

We could proceed analogously to Step 1 and modify \( f \) by increasing the value of \( f(u) \) so that it is strictly less than \( f(c) \), and strictly greater than \( f(h) \) for all \( h \in P \) such that \( h < c \), in which case the critical elements would not have changed, and now \( u \) would not be short-down-troubled. However, by increasing the value of \( f(u) \) in this way, we might cause \( u \) to become short-up-troubled, and so we do not modify \( f \) yet, but rather make the following additional observation. Suppose that there are \( x, y \in P \) such that \( u < x < y \) and \( f(x) < f(y) \). Let \( r \in P \) be such that \( r < c \). Then \( r < c < u < x < y \), and hence \( r < x < y \). Because \( r \) is not up-troubled, and hence \( f(r) < f(y) \). We now modify \( f \) by increasing \( f(u) \) so that it is strictly less than \( f(c) \), and strictly greater than \( f(h) \) for all \( h \in P \) such that \( h < c \), and strictly
less than \( f(i) \) for all \( i \in P \) such that there is some \( j \in P \) such that \( u \prec j \prec i \) and \( f(j) < f(i) \). We then see that the modified \( f \) is still a discrete Morse function, no critical elements have changed, and now \( u \) is not short-down-troubled and not short-up-troubled. The one remaining question is whether any element of \( P \) other than \( u \) has become short-up-troubled as a result of this modification of \( f \). The only possible elements of \( P \) that could become short-up-troubled as a result of modifying \( f(u) \) are elements \( s \in P \) for which there exist an element \( t \in P \) such that \( s \prec u \prec t \) or \( s \prec t \prec u \); however, it is seen that in either such case, increasing the value of \( f(u) \) could not make \( s \) become short-up-troubled if \( s \) is not already such prior to increasing \( f(u) \).

By recursion, we can modify \( f \) so that it has no short-down-troubled elements and no short-up-troubled elements, and has the same critical elements as it had originally.

Step 4: Similarly to Step 2, it can be proved that the modified \( f \) has no down-troubled elements, in addition to no up-troubled elements. Hence, the modified \( f \) satisfies Part 4, Part 1 and Part 2 of the lemma.

Step 5: Let \( a \in P \). We say that \( a \) is \textbf{general} with respect to \( f \) if \( f(a) \neq f(x) \) for all \( x \in P - \{a\} \).

As before, we proceed recursively, using the total order on \( P \) given in Step 1. Observe that \( f(P) \) is a finite set. Let \( a \in P \) be the least element of \( P \) with respect to the total order. Then modify \( f \) by increasing \( f(a) \) very slightly, in such a way that \( a \) is general after the modification, and that nothing in \( f(P) \) is between the original value of \( f(a) \) and the new value. It is then seen that if \( x, y \in P \) are such that \( f(x) < f(y) \) prior to the modification, it must still be the case that \( f(x) < f(y) \) after the modification. It follows that no elements of \( P \) can become up-troubled or down-troubled as a result of the modification, that the modified \( f \) is still a discrete Morse function, and that if an element of \( P \) is discrete-critical prior to the modification, then it remains so after the modification. Suppose that \( a \) is discrete-ordinary prior to the modification. Because \( a \) is minimal with respect to \( \prec \), then it must be the case that before the modification \( f(a) \geq f(b) \) for some \( b \in P \) such that \( a \prec b \), and hence \( b \) will continue to be discrete-ordinary after the modification. A similar argument shows that no other elements of \( P \) can change from discrete-ordinary to discrete-critical as a result of the modification.

Now suppose that \( f \) has been modified so that the first \( k - 1 \) elements of \( P \) in the total order are general, that the modified \( f \) is still a discrete Morse function, that the critical elements have not changed, and that no elements are up-troubled or down-troubled after the modification. Let \( e \in P \) be the \( k \)-th element of \( P \) in the total order. As before, we modify \( f \) by increasing \( f(e) \) very slightly, in such a way that \( e \) is general after the modification, and that nothing in \( f(P) \) is between the original value of \( f(e) \) and the new value. Once again \( e \) is general after the modification, the modified \( f \) is still a discrete Morse function, no elements of \( P \) can become up-troubled or down-troubled as a result of the modification, and if an element of \( P \) is discrete-critical prior to the modification, then it remains so after the modification. Suppose that \( e \) is discrete-ordinary prior to the modification. First, suppose that there is some \( x \in P \) such that \( x \prec e \) and \( f(x) \geq f(e) \) prior to
the modification. Because \( x < e \), then \( x \) is prior to \( e \) in the total order, and hence \( x \) is general. Therefore \( f(x) > f(e) \) prior to the modification, and this inequality will still hold after the modification, and hence \( e \) will remain discrete-ordinary. Second, suppose that there is some \( y \in P \) such that \( e < y \) and \( f(e) \geq f(y) \) prior to the modification. Then after the modification we will have \( f(e) > f(y) \), and hence \( e \) will remain discrete-ordinary.

By recursion, we can modify \( f \) so that all elements are general, the modified \( f \) is a discrete Morse function, it has no short-down-troubled elements and no short-up-troubled elements, and has the same critical elements as it had originally. It follows that all four properties of the lemma hold for the modified \( f \).

We are now ready for the proof of our theorem.

**Proof of Theorem 1.3.** We will show that an embedding with the desired property can be found for a single choice of \( \mathbb{R}^m \) and with respect to projection onto the \( x \)-axis. It will then follow immediately that an appropriate embedding can be found in \( \mathbb{R}^k \) for \( k > m \) with respect to projection onto the \( x \)-axis by using using the usual embedding of \( \mathbb{R}^m \) in \( \mathbb{R}^k \). Appropriate embeddings with respect to any other line in \( \mathbb{R}^k \) can be found by rotating and translating the original embedding.

Let \( g \) be the discrete Morse function on \( P \) obtained by applying Lemma 2.3 to \( f \). By Part 4 of the lemma, it will suffice to prove the theorem with \( f \) replaced by \( g \).

Suppose \( P \) has \( k \) elements. By Lemma 2.1 there is a map \( \psi : P \to \mathbb{R}^k \) such that \( \psi(P) \) spans an \((k - 1)\)-simplex, and for each vertex \( v \in V \), the projection of \( \psi(v) \) onto the \( x \)-axis equals \( g(v) \). Because \( \Delta(P) \) is a simplicial complex with \( k \) vertices, it can be thought of as a subcomplex of the \((k - 1)\)-simplex spanned by \( \psi(P) \). Hence we can think of \( \psi \) as a polyhedral embedding \( \phi : \Delta(P) \to \mathbb{R}^m \), where \( \phi(v) = \psi(v) \) for all \( v \in P \), where we think of \( P \) as the set of vertices of \( \Delta(P) \).

Lemma 2.3 (3) says that if \( a, b \in P \) and \( a < b \), then \( g(a) \neq g(b) \). It follows that if \( a, b \) are vertices of \( \Delta(P) \) that are joined by an edge, then \( g(a) \neq g(b) \), and hence the projection of \( \psi(a) \) onto the \( x \)-axis does not equal the projection of \( \psi(b) \) onto the \( x \)-axis. We can therefore define Banchhoff’s index at the vertices of \( \phi(\Delta(P)) \), and so we can apply the notion of polyhedral-critical and polyhedral-ordinary to these vertices.

Let \( b \in P \), so that \( b \) is a vertex of \( \Delta(P) \). We need to show that \( \phi(b) \) is polyhedral-critical with respect to projection onto the \( x \)-axis if and only if \( b \) is discrete-critical with respect to \( g \).

Let \( \xi \) denote the unit vector in the direction of the positive \( x \)-axis. Following [Ban67], we compute \( a(\phi(b), \xi) \) as follows. Let \( T \) denote the set of all simplices of \( \Delta(P) \) that contain \( \phi(b) \) as a vertex and for which projection onto the \( x \)-axis has maximal value at \( \phi(b) \). Then \( a(\phi(b), \xi) = \sum_{s \in T} (-1)^{\dim s} \). We can view this last formula from a different perspective. By the definition of \( \Delta(P) \), every simplex of \( \Delta(P) \) is a non-empty chain in \( P \). The choice of \( \phi \) states that the projection of \( \phi(a) \) onto the \( x \)-axis equals \( g(a) \) for all \( a \in P \). Hence, we see we can think of \( T \) as the set of all chains in \( P \) that contain \( b \), and on which \( g \) is maximal at \( b \). If \( C \) is a chain in
P, then the dimension of this chain when thought of as a simplex of $\Delta(P)$ is equal to $l(C)$. Therefore $a(\phi(b), \xi) = \sum_{C \subseteq T} (-1)^{|l(C)|}$.

Suppose that $b$ is discrete-critical with respect to $g$. Let $v \in P$ be such that $v < b$. If $v \prec b$, then $g(v) < g(b)$ because $b$ is discrete-critical with respect to $g$. Now suppose $v \not\prec b$. A basic property of posets implies that there is some $z \in P$ such that $v < z < b$. Because $b$ is discrete-critical with respect to $g$, then $g(z) < g(b)$. By Lemma 2.3 (1) we deduce that $g(v) < g(b)$. A similar argument shows that $g(b) < g(u)$ for any $u \in P$ such that $b < u$. Hence, the set $T$ consists precisely of all chains in $P_{\leq b}$ that contain $b$, so that $T = ch(b; b)$. Lemma 2.2 (1) implies that

$$a(\phi(b), \xi) = \sum_{C \subseteq T} (-1)^{|l(C)|} = \sum_{C \subseteq ch(b; b)} (-1)^{|l(C)|} = (-1)^{u(b)} \neq 0.$$ 

Therefore $\phi(b)$ is polyhedral-critical with respect to projection onto the $x$-axis.

Next, suppose that $b$ is discrete-ordinary with respect to $g$. Then by Lemma 1.2 either there is a single $h \in P$ such that $h \prec b$ and $g(h) \geq g(b)$, or there is a single $u \in P$ such that $b < u$ and $g(b) \geq f(u)$, but not both.

First, suppose that there is a single $h \in P$ such that $h \prec b$ and $g(h) \geq g(b)$. Then $g(b) < g(z)$ for all $z \in P$ such that $b < z$. By the same argument used above, we know that $g(b) < g(u)$ for any $u \in P$ such that $b < u$, and that $T \subseteq ch(b; b)$.

Let $c \in P$ be such that $c < b$ and $c \neq h$. If $c \not\prec h$, then the definition of discrete Morse functions implies that $g(c) < g(b)$. Now suppose $c \not\prec b$. Combining basic properties of posets with the fact that $P$ is two-wide, there is some $t \in P$ such that $t \neq h$ and $c < t < b$. Then $g(t) < g(b)$, again by the definition of discrete Morse functions, and it follows from Lemma 2.3 (1) that $g(c) < g(b)$.

Putting the above considerations together, we see that $h$ is the only element of $P_{\leq b}$ such that $g(h) > g(b)$. Hence $T = ch(b; h)$, and Lemma 2.2 (2) implies that

$$a(\phi(b), \xi) = \sum_{C \subseteq ch(b; h)} (-1)^{|l(C)|} = 0.$$ 

Therefore $\phi(b)$ is polyhedral-ordinary with respect to projection onto the $x$-axis.

Second, suppose that there is a single $u \in P$ such that $b < u$ and $g(b) \geq f(u)$. An argument similar to the previous case shows that $T = ch(u; b)$. Lemma 2.2 (3) implies that $a(\phi(b), \xi) = 0$, and hence $\phi(b)$ is polyhedral-ordinary with respect to projection onto the $x$-axis.

\[\square\]

\textbf{References}


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