MATH 241 Vector Calculus Spring 2016
Study Sheet for Final Exam

- This study sheet will not be allowed during the test.
- Books, notes and online resources will not be allowed during the test.
- Electronic devices (calculators, cell phones, tablets, laptops, etc.) will not be allowed during the test.

Topics

1. Double integrals and iterated integrals.
2. Polar coordinates.
3. Double integrals in polar coordinates.
4. Vector fields.
5. Divergence and curl.
6. Conservative vector fields.
7. Line integrals of functions and of vector fields.
8. Fundamental Theorem of Calculus for Line Integrals.
9. Path Independent Line Integrals.
10. Green's Theorem.
11. Sequences.
12. Series (convergence of series, telescoping series, geometric series, p-series).
13. Convergence tests for series (Divergence Test, Comparison Test, Limit Comparison Test, Integral Test, Alternating Series Test, Ratio Test).
14. Power series (interval of convergence and radius of convergence).
15. Differentiation and integration of power series.
16. Taylor series and Maclaurin Series.

Practice Problems from Stewart, Calculus Concepts and Contexts, 4th ed.

Section 12.1: 1, 3, 5, 9a

Section 12.2: 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 25, 27, 29, 31

Section 12.3: 1, 3, 5, 7, 9, 15, 17, 19, 21, 23, 25, 27, 29, 31, 41, 43, 45, 47, 49, 51

Appendix H.1: 1, 3, 5, 7, 9, 11

Section 12.4: 5, 7, 9, 11, 13, 15, 17, 19, 27, 29, 31

Section 13.1: 1, 3, 5, 11–14, 15–18, 29–32

Section 13.2: 1, 3, 5, 7, 9, 11, 13, 15, 19, 21

Section 13.3: 3, 5, 7, 9, 13, 15, 17

Section 13.4: 1, 3, 5, 7, 9, 11, 13

Section 13.5: 1, 3, 5, 7, 13, 15, 17

Section 8.1: 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 41, 43

Section 8.2: 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 37, 39, 49, 51, 53, 65

Section 8.3: 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31

Section 8.4: 3, 5, 7, 9, 13, 21, 23, 25, 27, 29, 31, 33, 37

Section 8.5: 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25

Section 8.7: 5, 7, 9, 11, 13, 15, 17, 39, 43, 45, 47, 49

Some Important Concepts and Formulas

1. Double Integrals over Rectangles

1. Let $f : R \to \mathbb{R}$ be a function defined on a rectangle $R = [a, b] \times [c, d]$ in $\mathbb{R}^2$. The **integral** of f over R is

$$\iint_R f(x, y)\,dA = \lim_{\substack{\max \Delta x_i \to 0 \\ \max \Delta y_j \to 0}} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A_{ij},$$

provided the limit exists, and is the same, for all choices of Riemann sums. If this limit exists, the function f is **integrable**.

2. Every continuous function is integrable on any rectangle.

2. Iterated Integrals over Rectangles

Let $f : R \to \mathbb{R}$ be a function defined on a rectangle $R = [a, b] \times [c, d]$ in $\mathbb{R}^2$. Suppose that f is continuous.

$$\iint_R f(x, y)\,dA = \int_a^b \int_c^d f(x, y)\,dy\,dx = \int_c^d \int_a^b f(x, y)\,dx\,dy.$$

3. Double Integrals and Iterated Integrals over General Regions

Let $f : D \to \mathbb{R}$ be a function defined on a closed bounded region D of $\mathbb{R}^2$. Suppose that f is continuous.

Type I

Suppose that the region D is given by inequalities of the form

$$a \le x \le b$$
$$g_1(x) \le y \le g_2(x).$$

Then

$$\iint_D f(x, y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\,dx.$$

Type II

Suppose that the region D is given by inequalities of the form

$$c \le y \le d$$
$$h_1(y) \le y \le h_2(y).$$

Then

$$\iint_D f(x, y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\,dx\,dy.$$

4. Basic Rules for Double Integrals

Let $f, g : D \to \mathbb{R}$ be functions defined on a closed bounded region D of $\mathbb{R}^2$, and let $k \in \mathbb{R}$. Suppose that f and g are integrable.

1. $$\iint_D [f(x,y) + g(x,y)]\, dA = \iint_D f(x,y)\, dA + \iint_D g(x,y)\, dA.$$

2. $$\iint_D [f(x,y) - g(x,y)]\, dA = \iint_D f(x,y)\, dA - \iint_D g(x,y)\, dA.$$

3. $$\iint_D k f(x,y)\, dA = k \iint_D f(x,y)\, dA.$$

4. $$\iint_D k\, dA = k \cdot \text{area}(D).$$

5. Breaking up the Region for Double Integrals

Let $f : D \to \mathbb{R}$ be a function defined on a closed bounded region D of $\mathbb{R}^2$. Suppose that f is integrable. Suppose that D is the union of two regions D_1 and D_2 that overlap at most on their boundaries.

$$\iint_D f(x,y)\, dA = \iint_{D_1} f(x,y)\, dA + \iint_{D_2} f(x,y)\, dA.$$

6. Basic Inequalities for Double Integrals

Let $f, g : D \to \mathbb{R}$ be functions defined on a closed bounded region D of $\mathbb{R}^2$. Suppose that f and g are integrable.

1. If $f(x,y) \geq 0$ on D, then $\displaystyle\iint_D f(x,y)\, dA \geq 0$.

2. If $f(x,y) \leq g(x,y)$ on D, then $\displaystyle\iint_D f(x,y)\, dA \leq \iint_D g(x,y)\, dA$.

3. If $m \leq f(x,y) \leq M$ on D, then $\displaystyle m \cdot \text{area}(D) \leq \iint_D f(x,y)\, dA \leq M \cdot \text{area}(D)$.

7. Polar Coordinates

Let (x, y) be the rectangular coordinates of a point in $\mathbb{R}^2$, and let (r, θ) be the polar coordinates of the same point.

1. $x = r\cos\theta$ and $y = r\sin\theta$.
2. $r = \sqrt{x^2 + y^2}$ and $\tan\theta = \frac{y}{x}$.

8. Double Integrals in Polar Coordinates

Let D be a region of $\mathbb{R}^2$ defined by inequalities of the form

$$\begin{gathered} \alpha \leq \theta \leq \beta \\ h_1(\theta) \leq r \leq h_2(\theta), \end{gathered}$$

and let $f : D \to \mathbb{R}$ be a function. Suppose that f is continuous.

$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

9. Vector Fields

1. Let $n \in \mathbb{N}$. A **vector field** on $\mathbb{R}^n$ is a function $\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^n$.
2. Let $E \subseteq \mathbb{R}^n$ be a subset. A **vector field** on E is a function $\boldsymbol{F} : E \to \mathbb{R}^n$.
3. A vector-field has the form

$$\boldsymbol{F}(x_1, x_2, \dots, x_n) = \begin{bmatrix} f_1(x_1, x_2, \dots, x_n) \\ f_2(x_1, x_2, \dots, x_n) \\ \vdots \\ f_n(x_1, x_2, \dots, x_n) \end{bmatrix}.$$

4. A vector field on $\mathbb{R}^3$ has the form

$$\boldsymbol{F}(x, y, y) = \begin{bmatrix} P(x,y,z) \\ Q(x,y,z) \\ R(x,y,z) \end{bmatrix}.$$

5. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. Then the gradient ∇f is a vector field on $\mathbb{R}^n$.

10. Divergence and Curl

Let $\boldsymbol{F}: \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field on $\mathbb{R}^3$. Suppose that $\boldsymbol{F}(x, y, z)$ is given by

$$\boldsymbol{F}(x, y, z) = \begin{bmatrix} P(x,y,z) \\ Q(x,y,z) \\ R(x,y,z) \end{bmatrix}.$$

Divergence of $\boldsymbol{F}$

$$\operatorname{div} \boldsymbol{F} = \nabla \cdot \boldsymbol{F}(x, y, z) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Curl of $\boldsymbol{F}$

$$\operatorname{curl} \boldsymbol{F} = \nabla \times \boldsymbol{F}(x, y, z) = \det \begin{bmatrix} \boldsymbol{i} & \frac{\partial}{\partial x} & P \\ \boldsymbol{j} & \frac{\partial}{\partial y} & Q \\ \boldsymbol{k} & \frac{\partial}{\partial z} & R \end{bmatrix} = \begin{bmatrix} \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \\ \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \end{bmatrix}.$$

11. Curl of the Gradient

Let $f: \mathbb{R}^3 \to \mathbb{R}$ be a function. Suppose that f has continuous second-order partial derivatives. Then $\operatorname{curl}(\nabla f) = \boldsymbol{0}$, which means that it is constantly zero for all (x, y, z).

12. Conservative Vector Fields

Let $E \subseteq \mathbb{R}^n$ be an open subset, and let $\boldsymbol{F}: E \to \mathbb{R}^n$ be a vector field on E. The vector field $\boldsymbol{F}$ is **conservative** if $\boldsymbol{F} = \nabla f$ for some some function $f: E \to \mathbb{R}$; the function f is called a **potential function** for $\boldsymbol{F}$.

13. When is a Vector Field Conservative

Let $E \subseteq \mathbb{R}^n$ be an open subset, and let $\boldsymbol{F}: E \to \mathbb{R}^n$ be a vector field on E. Suppose that E is a simply connected region of $\mathbb{R}^3$, and that the components of $\boldsymbol{F}$ have continuous partial derivatives. Then $\boldsymbol{F}$ is conservative if and only if $\operatorname{curl} \boldsymbol{F} = \boldsymbol{0}$.

14. Finding a Potential Function for a Conservative Vector Field

Let $E \subseteq \mathbb{R}^3$ be an open subset, and let $\boldsymbol{F}: E \to \mathbb{R}^3$ be a vector field on E. Suppose that $\boldsymbol{F}(x, y, z) = \begin{bmatrix} P(x,y,z) \\ Q(x,y,z) \\ R(x,y,z) \end{bmatrix}$, and that $\boldsymbol{F}$ is conservative. To find a function f such that $\boldsymbol{F} = \nabla f$, solve the three equations

$$\frac{\partial f}{\partial x} = P(x, y, z) \quad \text{and} \quad \frac{\partial f}{\partial y} = Q(x, y, z) \quad \text{and} \quad \frac{\partial f}{\partial z} = R(x, y, z)$$

by taking the antiderivative of one these equations with respect to the relevant variable, and then substitute the result into the other two equations.

15. Conservative Vector Fields on $\mathbb{R}^2$

Let $E \subseteq \mathbb{R}^2$ be a simply connected open subset, and let $\boldsymbol{F} : E \to \mathbb{R}^2$ be a vector field on E. Suppose that $\boldsymbol{F}(x, y) = \begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix}$. Define the vector field $\hat{\boldsymbol{F}} : E \times \mathbb{R} \to \mathbb{R}^3$ by the formula $\hat{\boldsymbol{F}}(x, y, z) = \begin{bmatrix} P(x,y) \\ Q(x,y) \\ 0 \end{bmatrix}$. Then $\boldsymbol{F}(x, y)$ is conservative if and only if $\hat{\boldsymbol{F}}(x, y, z)$ is conservative if and only if $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$.

16. Line Integrals of Functions

Let $E \subseteq \mathbb{R}^2$ be an open subset, and let $r : E \to \mathbb{R}$ be a function. Let C be a smooth curve in E given by a vector-valued function $\mathbf{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ defined on the interval $[a, b]$. There are three types of line integrals of f along C.

Line Integral with Respect to Arc Length

$$\int_C f(x, y)\, ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\, dt.$$

Line Integral with Respect to x

$$\int_C f(x, y)\, dx = \int_a^b f(\mathbf{r}(t))x'(t)\, dt.$$

Line Integral with Respect to y

$$\int_C f(x, y)\, dy = \int_a^b f(\mathbf{r}(t))y'(t)\, dt.$$

17. Line Integrals of Vector Fields

Let $E \subseteq \mathbb{R}^2$ be an open subset, and let $\boldsymbol{F} : E \to \mathbb{R}^2$ be a vector field on E. Suppose that $\boldsymbol{F}(x, y) = \begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix}$. Let C be a smooth curve in E given by a vector-valued function $\boldsymbol{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ defined on the interval $[a, b]$. Let $\boldsymbol{T}(t)$ be the unit tangent vector to $\boldsymbol{r}(t)$. There are two types of line integrals of $\boldsymbol{F}$ along C.

Tangential Line Integral

$$\begin{aligned} \int_C \boldsymbol{F} \cdot \boldsymbol{T}\, ds = \int_C \boldsymbol{F} \cdot d\boldsymbol{r} &= \int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)\, dt \\ &= \int_C P(x, y)\, dx + Q(x, y)\, dy. \end{aligned}$$

Normal Line Integral

$$\int_C \boldsymbol{F} \cdot \boldsymbol{n}\, ds = \int_C -Q(x, y)\, dx + P(x, y)\, dy.$$

18. Fundamental Theorem of Calculus for Line Integrals

Suppose $E \subseteq \mathbb{R}^2$ is an open subset, and suppose $f : E \to \mathbb{R}$ be a function. Suppose that f has continuous partial derivatives. Suppose C is a smooth curve in E given by a vector-valued function $\boldsymbol{r}(t)$ defined on the interval $[a, b]$. Then

$$\int_C \nabla f \cdot d\boldsymbol{r} = f(\boldsymbol{r}(b)) - f(\boldsymbol{r}(a)).$$

19. Paths and Closed Curves

1. A **path** in $\mathbb{R}^2$ is a continuous function of the form $\boldsymbol{r} : [a, b] \to \mathbb{R}^2$, for some closed bounded interval $[a, b]$. Similarly for paths in $\mathbb{R}^3$.

2. If $\boldsymbol{r} : [a, b] \to \mathbb{R}^2$ is a path, the **initial point** of $\boldsymbol{r}$ is $\boldsymbol{r}(a)$, and the **terminal point** of $\boldsymbol{r}$ is $\boldsymbol{r}(b)$.

3. A **closed curve** in $\mathbb{R}^2$ is a path $\boldsymbol{r} : [a, b] \to \mathbb{R}^2$ such that $\boldsymbol{r}(a) = \boldsymbol{r}(b)$. Similarly for closed curves in $\mathbb{R}^3$.

4. A **simple closed curve** in $\mathbb{R}^2$ is a closed curve is a path $\boldsymbol{r} : [a, b] \to \mathbb{R}^2$ such that $\boldsymbol{r}$ does not intersect itself on on $[a, b)$.

5. A simple closed curve in $\mathbb{R}^2$ is **positively oriented** if it is traversed in the counterclockwise direction.

6. Let $E \subseteq \mathbb{R}^2$ be an open subset, and let $f : E \to \mathbb{R}$ be a function. Let C be a smooth simple closed curve in E given by a vector-valued function $\boldsymbol{r}(t)$ defined on the interval $[a, b]$. Because C is a simple closed curve, the line integral of f along C is denoted

$$\oint_C f(x, y)\, ds.$$

20. Green's Theorem

Let $E \subseteq \mathbb{R}^2$ be an open subset, and let $\mathbf{F}\colon E \to \mathbb{R}^2$ be a vector field on E. Suppose that $\mathbf{F}(x,y) = \begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix}$. Suppose that P and Q have continuous partial derivatives. Let C be a positively oriented, piecewise smooth, simple closed curve in E given by a vector-valued function $\mathbf{r}(t)$ defined on the interval $[a,b]$. Let D be the region bounded by C.

Curl Version

$$\iint_D \operatorname{curl}\mathbf{F}\cdot\mathbf{k}\,dA = \oint_C \mathbf{F}\cdot d\mathbf{r}$$

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \oint_C P(x,y)\,dx + Q(x,y)\,dy.$$

Divergence Version

$$\iint_D \operatorname{div}\mathbf{F}\,dA = \oint_C \mathbf{F}\cdot\mathbf{n}\,ds$$

$$\iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA = \oint_C -Q(x,y)\,dx + P(x,y)\,dy.$$

21. Sequences

1. A **sequence** of real numbers is a collection of real numbers of which there is a first, a second, a third and so on, with one real number for each element of $\mathbb{N}$. A sequence is written $a_1, a_2, a_3, \ldots$, and also $\{a_n\}_{n=1}^{\infty}$.
2. The index n of a sequence could start at any number, not just 1.
3. In mathematical usage, the terms "sequence" and "series" mean different things, and should be used according to their precise meanings.
4. As sequence can be defined **explicitly**, which means that the sequence is given by a formula for a_n in terms of n, or **recursively**, which means that the sequence is given by specifying a_1 together with a formula for a_{n+1} in terms of a_n.

22. Sequences: Limits

1. Let $\left\{a_n\right\}_{n=1}^{\infty}$ be a sequence, and let $L \in \mathbb{R}$. The number L is the **limit** of $\left\{a_n\right\}_{n=1}^{\infty}$, written

$$\lim_{n\to\infty} a_n = L,$$

if the value of a_n gets closer and closer to a number L as the value of n gets larger and larger. If $\lim_{n\to\infty} a_n = L$, the sequence $\left\{a_n\right\}_{n=1}^{\infty}$ **converges** to L. If $\left\{a_n\right\}_{n=1}^{\infty}$ converges to some real number, the sequence $\left\{a_n\right\}_{n=1}^{\infty}$ is **convergent**; otherwise $\left\{a_n\right\}_{n=1}^{\infty}$ is **divergent**.

2. The above definition, and in particular the use of the phrase "gets closer and closer," is informal. A rigorous definition of limits will be seen in a Real Analysis course.

3. If a sequence has a limit, the limit is unique.

4. Let $\left\{a_n\right\}_{n=1}^{\infty}$ be a sequence. Let $f : [1, \infty) \to \mathbb{R}$ be a function such that $f(n) = a_n$ for all n in $\mathbb{N}$. If $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} a_n = L$.

23. Sequences: Basic Limits

1. $\lim_{n\to\infty} \frac{1}{n} = 0$.

2.

$$\lim_{n\to\infty} r^n = \begin{cases} 0, & \text{if } |r| < 1 \\ 1, & \text{if } r = 1 \\ \text{does not exist}, & \text{otherwise.} \end{cases}$$

24. Sequences: Properties of Limits

Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ be sequences, and let $k \in \mathbb{R}$. Suppose that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are convergent.

1. $\{a_n + b_n\}_{n=1}^{\infty}$ is convergent and $\lim\limits_{n\to\infty}(a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$.
2. $\{a_n - b_n\}_{n=1}^{\infty}$ is convergent and $\lim\limits_{n\to\infty}(a_n - b_n) = \lim\limits_{n\to\infty} a_n - \lim\limits_{n\to\infty} b_n$.
3. $\{ka_n\}_{n=1}^{\infty}$ is convergent and $\lim\limits_{n\to\infty} ka_n = k\lim\limits_{n\to\infty} a_n$.
4. $\{a_n b_n\}_{n=1}^{\infty}$ is convergent and $\lim\limits_{n\to\infty} a_n b_n = [\lim\limits_{n\to\infty} a_n] \cdot [\lim\limits_{n\to\infty} b_n]$.
5. If $\lim\limits_{n\to\infty} b_n \neq 0$, then $\left\{\frac{a_n}{b_n}\right\}_{n=1}^{\infty}$ is convergent and $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$.
6. If $f(x)$ is a continuous function, then $\lim\limits_{n\to\infty} f(a_n) = f(\lim\limits_{n\to\infty} a_n)$.
7. If $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $\lim\limits_{n\to\infty} a_n \leq \lim\limits_{n\to\infty} b_n$.
8. (Squeeze Theorem) If $a_n \leq c_n \leq b_n$ for all $n \in \mathbb{N}$, and if $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n$, then $\{c_n\}_{n=1}^{\infty}$ is convergent and $\lim\limits_{n\to\infty} c_n = \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n$.

25. Series

1. A **series** of real numbers is a formal sum of a sequence of real numbers, written

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots .$$

2. The index n of a series could start at any number, not just 1.

26. Series: Convergence

Let $\sum_{n=1}^{\infty} a_n$ be a series.

1. For each $k \in \mathbb{N}$, the k^{th} **partial sum** of $\sum_{n=1}^{\infty} a_n$, denoted s_k, is defined by

$$s_k = \sum_{i=1}^{k} a_i = a_1 + a_2 + \cdots + a_k.$$

2. The **sequence of partial sums** of $\sum_{n=1}^{\infty} a_n$ is the sequence $\left\{s_n\right\}_{n=1}^{\infty}$.

3. Let $L \in \mathbb{R}$. The number L is the **sum** of $\sum_{n=1}^{\infty} a_n$, written

$$\sum_{n=1}^{\infty} a_n = L,$$

   if $\lim_{n\to\infty} s_n = L$. If $\sum_{n=1}^{\infty} a_n = L$, the series $\sum_{n=1}^{\infty} a_n$ **converges** to L. If $\sum_{n=1}^{\infty} a_n$ converges to some real number, the series $\sum_{n=1}^{\infty} a_n$ is **convergent**; otherwise $\sum_{n=1}^{\infty} a_n$ is **divergent**.

4. If a series has a sum, the sum is unique.

5. Changing or deleting a finite numbers of terms in a series will not affect whether the series is convergent or divergent (though it might change the sum of the series if the series is convergent).

27. Harmonic Series

1. The **harmonic series** is the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots.$$

2. The harmonic series is divergent.

28. Geometric Series

1. A **geometric series** is any series of the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots,$$

   where $a, r \in \mathbb{R}$.

2. A geometric series converges to $\dfrac{a}{1-r}$ if $|r| < 1$, and is divergent if $|r| \geq 1$.

29. Series: Properties

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series, and let $k \in \mathbb{R}$. Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent.

1. $\sum_{n=1}^{\infty} (a_n + b_n)$ is convergent and $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.
2. $\sum_{n=1}^{\infty} (a_n - b_n)$ is convergent and $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$.
3. $\sum_{n=1}^{\infty} ka_n$ is convergent and $\sum_{n=1}^{\infty} ka_n = k \sum_{n=1}^{\infty} a_n$.

30. Divergence Test

Let $\sum_{n=1}^{\infty} a_n$ be a series.

1. If $\lim_{n\to\infty} a_n \neq 0$, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.
2. **Caution**: If $\lim_{n\to\infty} a_n = 0$, you CANNOT conclude that the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

31. Integral Test

Let $\sum_{n=1}^{\infty} a_n$ be a series, and let $f : [1, \infty) \to \mathbb{R}$ be function that satisfies the following four properties:

(1) $f(n) = a_n$ for all n.

(2) $f(x)$ is continuous on $[1, \infty)$.

(3) $f(x) > 0$ on $[1, \infty)$.

(4) $f(x)$ is decreasing on $[1, \infty)$.

Then $\sum_{n=1}^{\infty} a_n$ is convergent if and only if $\displaystyle\int_1^{\infty} f(x)\,dx$ is convergent.

32. p-Series

1. A ***p*-series** is any series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots,$$

   where $p \in \mathbb{R}$.

2. A p-series is convergent if $p > 1$, and is divergent if $p \leq 1$.

33. Comparison Test

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series. Suppose that $a_n \geq 0$ and $b_n \geq 0$ for all $n \in \mathbb{N}$. Suppose that $a_n \leq b_n$ for all $n \in \mathbb{N}$.

1. If $\displaystyle\sum_{n=1}^{\infty} b_n$ is convergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.
2. If $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent, then $\displaystyle\sum_{n=1}^{\infty} b_n$ is divergent.
3. **Caution**: If $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent or if $\displaystyle\sum_{n=1}^{\infty} b_n$ is divergent, you CANNOT conclude anything about the other series by the Comparison Test.

34. Limit Comparison Test

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series. Suppose that $a_n \geq 0$ and $b_n \geq 0$ for all $n \in \mathbb{N}$. Suppose that

$$\lim_{n\to\infty} \frac{b_n}{a_n} = L,$$

for some $L \in \mathbb{R}$ or $L = \infty$.

1. Suppose that $0 < L < \infty$. Then either both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent, or both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are divergent.
2. Suppose that $L = 0$. If $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent, then $\displaystyle\sum_{n=1}^{\infty} b_n$ is convergent.
3. Suppose that $L = \infty$. If $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent, then $\displaystyle\sum_{n=1}^{\infty} b_n$ is divergent.

35. Alternating Series

An **alternating series** is any series of the form

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n \qquad \text{or} \qquad \sum_{n=1}^{\infty} (-1)^n a_n,$$

where $a_n > 0$ for all $n \in \mathbb{N}$.

36. Alternating Series Test

Let $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ be an alternating series, where $a_n > 0$ for all $n \in \mathbb{N}$.

1. Suppose that the alternating series satisfies the following two properties:

(a) the sequence $\{a_n\}_{n=1}^{\infty}$ is decreasing.

(b) $\lim_{n\to\infty} a_n = 0$.

Then the alternating series is convergent.

2. The same result holds for alternating series of the form $\sum_{n=1}^{\infty} (-1)^n a_n$.

37. Remainder Estimate for the Alternating Series Test

Let $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ be an alternating series, where $a_n > 0$ for all $n \in \mathbb{N}$. Let $m \in \mathbb{N}$.

1. The m^{th} **remainder** of the alternating series, denoted R_m, is defined by

$$R_m = \sum_{n=1}^{\infty}(-1)^{n-1} a_n - s_m = \sum_{n=m+1}^{\infty} (-1)^n a_n.$$

2. Suppose that the alternating series satisfies the hypotheses of the Alternating Series Test, and hence is convergent. Then $|R_m| \leq a_{m+1}$.

3. The same result holds for alternating series of the form $\sum_{n=1}^{\infty} (-1)^n a_n$.

38. Absolute Convergence and Conditional Convergence

Let $\sum_{n=1}^{\infty} a_n$ be a series.

1. The series $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ is convergent.

2. The series $\sum_{n=1}^{\infty} a_n$ is **conditionally convergent** if $\sum_{n=1}^{\infty} a_n$ is convergent but not absolutely convergent.

3. If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

4. Any series is either absolutely convergent, conditionally convergent or divergent.

39. Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series. Suppose that $a_n \neq 0$ for all $n \in \mathbb{N}$. Suppose that

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L,$$

for some $L \in \mathbb{R}$ or $L = \infty$.

1. If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
2. If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.
3. **Caution**: If $L = 1$, you CANNOT conclude conclude that $\sum_{n=1}^{\infty} a_n$ is either convergent or divergent by the Ratio Test.

40. Power Series

1. A **power series** is any series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots,$$

   where $a, c_0, c_1, c_2, \cdots \in \mathbb{R}$.

2. If $a = 0$, a power series has the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots.$$

3. The numbers $c_0, c_1, c_2, \cdots$ are the **coefficients** of the power series.

41. Interval of Convergence and Radius of Convergence of Power Series

1. Let $\sum_{n=0}^{\infty} c_n(x-a)^n$ be a power series. Then precisely one of the following happens:

(1) The series is absolutely convergent for all real numbers x, in which case $R = \infty$.

(2) The series is convergent only for $x = a$, in which case $R = 0$.

(3) There is some positive number R such that the series is absolutely convergent for all $|x-a| < R$, and the series is divergent for all $|x-a| > R$.

2. The **radius of convergence** of the power series is R, which is either a real number or ∞.

3. The **interval of convergence** of the power series is set of all numbers x at which the power series is convergent.

4. (1) If $R = \infty$, the interval of convergence is $(-\infty, \infty)$.

(2) If $R = 0$, the interval of convergence is $[a, a]$.

(3) If $0 < R < \infty$, the the interval of convergence is one of $(a-R, a+R)$, or $(a-R, a+R]$, or $[a-R, a+R)$ or $[a-R, a+R]$.

5. To find the interval of convergence and radius of convergence, a method that often works is to use the Ratio Test, which leads to finding the radius convergence, and then, if $0 < R < \infty$, to use other convergence tests to find out convergence or divergence at the endpoints of the interval of convergence.

42. Representing a Function as a Power Series

1. Let $E \subseteq \mathbb{R}$ be a subset, let $f : E \to \mathbb{R}$ be a function, and let $\sum_{n=0}^{\infty} c_n(x-a)^n$ be a power series. The function f is **represented** by $\sum_{n=0}^{\infty} c_n(x-a)^n$ if the following three properties hold:

(1) The radius of convergence of $\sum_{n=0}^{\infty} c_n(x-a)^n$ is positive.

(2) The interval of convergence of $\sum_{n=0}^{\infty} c_n(x-a)^n$ is a subset of E.

(3) $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ for all x in the interval of convergence.

2. **Caution**: If f is represented by $\sum_{n=0}^{\infty} c_n(x-a)^n$, it is not necessarily the case that the interval of convergence of $\sum_{n=0}^{\infty} c_n(x-a)^n$ is all of E.

3. Not every function is represented by a power series.

4. If a function is represented by a power series, the power series is unique.

43. Differentiation and Integration of Power Series

Let $E \subseteq \mathbb{R}$ be a subset, let $f : E \to \mathbb{R}$ be a function, and let $\sum_{n=0}^{\infty} c_n(x-a)^n$ be a power series. Suppose that the function $f(x)$ is represented by $\sum_{n=0}^{\infty} c_n(x-a)^n$. Let R be the radius of convergence of $\sum_{n=0}^{\infty} c_n(x-a)^n$.

1. The power series $\displaystyle\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$ has radius of convergence R, and $f'(x) = \displaystyle\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$ for all $x \in (a-R, a+R)$.

2. The power series $= \displaystyle\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$ has radius of convergence R, and $\displaystyle\int f(x)\,dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$ for all $x \in (a-R, a+R)$.

3. **Caution**: For any particular function $f(x)$, it might be that the above power series are convergent on the endpoints of the interval $(a-R, a+R)$, and it might be that $f'(x)$ or $\int f(x)\,dx$ equals the power series at the endpoints, but that needs to be verified in each case.

44. Taylor Series and Maclaurin Series

Let $I \subseteq \mathbb{R}$ be an open interval, let $f : I \to \mathbb{R}$ be a function, and let $a \in I$. Suppose that f is infinitely differentiable.

1. The **Taylor series** of f centered at a is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots.$$

2. Suppose that $0 \in I$. The **Maclaurin series** of f is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots.$$

3. **Caution**: The Taylor series and Maclaurin series of a function do not always equal the function.

45. Taylor Series of Some Standard Functions

The following equalities hold for all $x \in \mathbb{R}$.

1.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

2.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

3.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

Basic Rules for Derivatives

1. $[f(x) + g(x)]' = f'(x) + g'(x)$

2. $[f(x) - g(x)]' = f'(x) - g'(x)$

3. $[cf(x)]' = cf'(x)$

4. $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$

5. $\left[\dfrac{f(x)}{g(x)}\right]' = \dfrac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

6. $[f(g(x))]' = f'(g(x))g'(x)$

Basic Derivatives

1. $(c)' = 0$

2. $(x)' = 1$

3. $(x^r)' = rx^{r-1}$, for any real number r

4. $(e^x)' = e^x$

5. $(a^x)' = a^x \ln a$

6. $(\ln x)' = \dfrac{1}{x}$

7. $(\ln |x|)' = \dfrac{1}{x}$

8. $(\log_a x)' = \dfrac{1}{\ln a}\dfrac{1}{x}$

9. $(\sin x)' = \cos x$

10. $(\cos x)' = -\sin x$

11. $(\tan x)' = \sec^2 x$

12. $(\sec x)' = \sec x \tan x$

13. $(\csc x)' = -\csc x \cot x$

14. $(\cot x)' = -\csc^2 x$

15. $(\arcsin x)' = \dfrac{1}{\sqrt{1 - x^2}}$

16. $(\arccos x)' = -\dfrac{1}{\sqrt{1 - x^2}}$

17. $(\arctan x)' = \dfrac{1}{1 + x^2}$

18. $(\operatorname{arcsec} x)' = \dfrac{1}{|x|\sqrt{x^2 - 1}}$

19. $(\operatorname{arccsc} x)' = -\dfrac{1}{|x|\sqrt{x^2 - 1}}$

20. $(\operatorname{arccot} x)' = -\dfrac{1}{1 + x^2}$

Basic Rules for Indefinite Integrals

1. $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$

2. $\int [f(x) - g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$

3. $\int c\,f(x)\,dx = c \int f(x)\,dx$

Basic Indefinite Integrals

1. $\int 1\,dx = x + C$

2. $\int x^r\,dx = \frac{x^{r+1}}{r+1} + C \quad \text{when } r \neq -1$

3. $\int \frac{1}{x}\,dx = \ln|x| + C$

4. $\int e^x\,dx = e^x + C$

5. $\int a^x\,dx = \frac{a^x}{\ln a} + C$

6. $\int \sin x\,dx = -\cos x + C$

7. $\int \cos x\,dx = \sin x + C$

8. $\int \sec^2 x\,dx = \tan x + C$

9. $\int \sec x \tan x\,dx = \sec x + C$

10. $\int \csc^2 x\,dx = -\cot x + C$

11. $\int \csc x \cot x\,dx = -\csc x + C$

12. $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$

13. $\int \frac{1}{1+x^2}\,dx = \arctan x + C$

14. $\int \frac{1}{|x|\sqrt{x^2-1}}\,dx = \operatorname{arcsec} x + C$