MATH 241 Vector Calculus Spring 2016
Study Sheet for Midterm Exam

- This study sheet will not be allowed during the test.
- Books, notes and online resources will not be allowed during the test.
- Electronic devices (calculators, cell phones, tablets, laptops, etc.) will not be allowed during the test.

Topics

1. Cross product.
2. Lines and planes in $\mathbb{R}^3$.
3. Single-variable vector-valued functions.
4. Derivatives and integrals of single-variable vector-valued functions.
5. Arc length.
6. Curvature.
7. Level curves for multivariable real-valued functions.
8. Partial derivatives of multivariable real-valued functions.
9. Derivative of multivariable vector-valued functions.
10. Jacobian determinant.
11. Chain rule for multivariable vector-valued functions.
12. Gradient.
13. Tangent planes and normal lines.
14. Directional derivatives.
15. Lagrange Multipliers.

Practice Problems from Stewart, Calculus Concepts and Contexts, 4th ed.

Section 9.3: 3, 5, 7, 9, 11, 15, 17, 19, 21, 23, 43.

Section 9.4: 1, 3, 7, 9, 11, 13, 15, 17, 19, 21, 23, 27, 29, 31.

Section 9.5: 1, 3, 5, 7, 9, 11, 13, 17, 19, 21, 23, 25, 27, 31, 37, 39, 41, 47, 49, 51.

Section 10.1: 1, 3, 5, 25, 27, 43.

Section 10.2: 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 31, 33, 35, 37, 39.

Section 10.3: 1, 3, 5, 17b, 19b, 21, 23.

Section 11.1: 1, 5, 7, 9, 11, 19, 21, 23, 25, 29.

Section 11.3: 1, 3, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39, 41, 43, 45, 47, 51, 53, 55, 57, 59, 61, 63, 65, 67, 69, 71, 75, 79, 81, 83.

Handout Section 31.2: 1, 2, 3, 4, 5, 6, 7.

Handout Section 31.4: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14.

Section 11.4: 1, 3, 5.

Section 11.6: 7, 9, 11, 13, 15, 17, 19, 21, 23, 27, 31, 39, 41, 43, 47, 49, 51, 53.

Section 11.8: 1, 3, 5, 7, 9, 11, 13, 15, 17, 27, 29, 31, 37, 39.

Some Important Concepts and Formulas

1. **Cross Product**

   Let $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. The **cross product** of **a** and **b** is defined by

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix} = \det \begin{bmatrix} \mathbf{i} & a_1 & b_1 \\ \mathbf{j} & a_2 & b_2 \\ \mathbf{k} & a_3 & b_3 \end{bmatrix}.$$

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2. **Properties of Cross Product**

   Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$, and let $s \in \mathbb{R}$.

   1. $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$.

   2. $(s\mathbf{a}) \times \mathbf{b} = s(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (s\mathbf{b})$.

   3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$.

   4. $(\mathbf{b} + \mathbf{c}) \times \mathbf{a} = \mathbf{b} \times \mathbf{a} + \mathbf{c} \times \mathbf{a}$.

   5. $\mathbf{0} \times \mathbf{a} = \mathbf{0}$.

   6. $\mathbf{a} \times (s\mathbf{a}) = \mathbf{0}$.

   7. $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$ and $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 0$.

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3. **Geometry of the Cross Product**

   Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$. Let θ be the angle between **a** and **b**.

   1. If **a** and **b** are non-zero and not parallel, then $\mathbf{a} \times \mathbf{b} = (|\mathbf{a}||\mathbf{b}| \sin\theta)\mathbf{n}$, where **n** is the unique unit vector in $\mathbb{R}^3$ that is perpendicular to both **a** and **b** and is in the direction given by the right hand rule.

   2. $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$.

   3. The area of the parallelogram formed by **a** and **b** is $|\mathbf{a} \times \mathbf{b}|$.

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4. **Scalar Triple Product**

Let $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$.

1. The **scalar triple product** of **a**, **b** and **c** is defined by

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.$$

2.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = -\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) = -\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}) = -\mathbf{c} \cdot (\mathbf{b} \times \mathbf{a}).$$

3. The volume of the parallelepiped formed by **a**, **b** and **c** is $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$.

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5. **Lines in $\mathbb{R}^3$**

Let $\mathbf{r}_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. The equation of the line through $\mathbf{r}_0$ and in the direction of **v** is given in the following three ways.

Vector Equation

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}.$$

Parametric Equations

$$\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct. \end{aligned}$$

Symmetric Equations

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

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6. **Planes in $\mathbb{R}^3$**

Let $\mathbf{r}_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$ and $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. The equation of the line through $\mathbf{r}_0$ and normal to **n** is given in the following three ways.

Vector Equation

$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0.$$

Scalar Equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Linear Equation

$$ax + by + cz + d = 0.$$

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7. **Single-Variable Vector-Valued Functions**

   1. A **single-variable vector-valued function** is a function $\mathbf{r} : \mathbb{R} \to \mathbb{R}^m$ for some $m \in \mathbb{N}$ such that $m \geq 2$.

   2. A single-variable vector-valued function has the form

$$\mathbf{r}(t) = \begin{bmatrix} r_1(t) \\ r_2(t) \\ \vdots \\ r_m(t) \end{bmatrix}.$$

   3. A function $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$ has the form

$$\mathbf{r}(t) = \begin{bmatrix} f(t) \\ g(t) \\ h(t) \end{bmatrix}.$$

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8. **Single-Variable Vector-Valued Functions: Limits**

   Let $\mathbf{r}(t) = \begin{bmatrix} f(t) \\ g(t) \\ h(t) \end{bmatrix}$ be a single-variable vector-valued function, and let $c \in \mathbb{R}$.

$$\lim_{t \to c} \mathbf{r}(t) = \begin{bmatrix} \lim_{t \to c} f(t) \\ \lim_{t \to c} g(t) \\ \lim_{t \to c} h(t) \end{bmatrix}.$$

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9. **Single-Variable Vector-Valued Functions: Derivatives**

   Let $\mathbf{r}(t)$ be a single-variable vector-valued function defined on an open interval.

   1. The derivative of $\mathbf{r}(t)$, denoted $\mathbf{r}'(t)$, is the function defined by

$$\mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h},$$

   for those values of t for which the limit exists.

   2. The function $\mathbf{r}(t)$ is **differentiable** if $\mathbf{r}'(t)$ is defined for all values of t.

   3. If $\mathbf{r}(t) = \begin{bmatrix} f(t) \\ g(t) \\ h(t) \end{bmatrix}$, then $\mathbf{r}'(t) = \begin{bmatrix} f'(t) \\ g'(t) \\ h'(t) \end{bmatrix}$.

   4. The unit tangent vector to $\mathbf{r}(t)$, denoted $\mathbf{T}(t)$, is defined by $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$, for those values of t for which $\mathbf{r}'(t) \neq 0$.

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10. **Single-Variable Vector-Valued Functions: Properties of Derivatives**

Let $\mathbf{r}(t)$ and $\mathbf{s}(t)$ be a single-variable vector-valued function, let $f(t)$ be a real-valued function, and let $c \in \mathbb{R}$. Suppose that $\mathbf{r}(t)$ and $\mathbf{s}(t)$ are differentiable.

1. $[\mathbf{r}(t) + \mathbf{s}(t)]' = \mathbf{r}'(t) + \mathbf{s}'(t)$.

2. $[\mathbf{r}(t) - \mathbf{s}(t)]' = \mathbf{r}'(t) - \mathbf{s}'(t)$.

3. $[c\mathbf{r}(t)]' = c\mathbf{r}'(t)$.

4. $[f(t)\mathbf{r}(t)]' = f'(t)\mathbf{r}(t) + f(t)\mathbf{r}'(t)$.

5. $[\mathbf{r}(t) \cdot \mathbf{s}(t)]' = \mathbf{r}'(t) \cdot \mathbf{s}(t) + \mathbf{r}(t) \cdot \mathbf{s}'(t)$.

6. $[\mathbf{r}(t) \times \mathbf{s}(t)]' = \mathbf{r}'(t) \times \mathbf{s}(t) + \mathbf{r}(t) \times \mathbf{s}'(t)$.

7. $[\mathbf{r}(f(t))]' = \mathbf{r}'(f(t))f'(t)$.

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11. **Single-Variable Vector-Valued Functions: Integrals**

Let $\mathbf{r}(t)$ be a single-variable vector-valued function.

1. If $\mathbf{r}(t) = \begin{bmatrix} f(t) \\ g(t) \\ h(t) \end{bmatrix}$, the integral of $\mathbf{r}(t)$ from a to b is

$$\int_a^b \mathbf{r}(t)\, dt = \begin{bmatrix} \int_a^b f(t)\, dt \\ \int_a^b g(t)\, dt \\ \int_a^b h(t)\, dt \end{bmatrix},$$

provided the three integrals exist.

2. **Fundamental Theorem of Calculus—Version II**: If $\mathbf{R}(t)$ is an antiderivative of $\mathbf{r}(t)$, then

$$\int_a^b \mathbf{r}(t)\, dt = \mathbf{R}(b) - \mathbf{R}(a).$$

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12. **Arc Length**

Let $\mathbf{r}(t)$ be a single-variable vector-valued function. Suppose that $\mathbf{r}(t)$ is differentiable. The arc length of $\mathbf{r}(t)$ from a to b is

$$L = \int_a^b |\mathbf{r}'(t)|\, dt.$$

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13. **Curvature**

Let $\mathbf{r}(t)$ be a single-variable vector-valued function.

1. The function $\mathbf{r}(t)$ is **regular** if $\mathbf{r}'(t)$ is continuous and $\mathbf{r}'(t) \neq 0$ for all t.
2. Suppose that $\mathbf{r}(t)$ is regular. The curvature of $\mathbf{r}(t)$, denoted $\kappa(t)$, is defined by

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.$$

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14. **Partial Derivatives of a Multivariable Real-Valued Function**

1. Let U be an open region in $\mathbb{R}^2$, and let $f : U \to \mathbb{R}$ be a function. The **partial derivatives** of $f(x, y)$ with respect to x and y respectively, denoted $f_x(x, y)$ and $f_y(x, y)$, are the functions given by

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} \quad \text{and} \quad f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h},$$

which are defined at all points for which the limit exists.
2. If g is a multivariable real-valued function of n of variables, there are n partial derivatives of g, each of which is obtained by considering all but one of the variables of g as constants, and taking the derivative with respect to the variable not being considered as a constant.

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15. **Notation for Partial Derivatives**

Let $z = f(x, y)$ be a function.

$$f_x(x, y), \quad f_1(x, y), \quad D_x f(x, y), \quad D_1 f(x, y), \quad \frac{\partial f}{\partial x}, \quad \frac{\partial z}{\partial x}, \quad \frac{\partial}{\partial x} f(x, y).$$

$$f_y(x, y), \quad f_2(x, y), \quad D_y f(x, y), \quad D_2 f(x, y), \quad \frac{\partial f}{\partial y}, \quad \frac{\partial z}{\partial y}, \quad \frac{\partial}{\partial y} f(x, y).$$

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16. **Notation for Second Partial Derivatives**

Let $z = f(x, y)$ be a function.

$$f_{xx}(x, y), \quad f_{11}(x, y), \quad \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 z}{\partial x^2}.$$

$$f_{yy}(x, y), \quad f_{22}(x, y), \quad \frac{\partial^2 f}{\partial y^2}, \quad \frac{\partial^2 z}{\partial y^2}.$$

$$f_{xy}(x, y), \quad f_{12}(x, y), \quad \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 z}{\partial y \partial x}.$$

$$f_{yx}(x, y), \quad f_{21}(x, y), \quad \frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial^2 z}{\partial x \partial y}.$$

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17. **Clairaut's Theorem**

Let $f(x, y)$ be a function defined on an open disk in $\mathbb{R}^2$. If $f_{xy}(x, y)$ and $f_{yx}(x, y)$ are both continuous on the disk, then $f_{xy}(x, y) = f_{yx}(x, y)$ for all (x, y) in the disk.

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18. **Multivariable Vector-Valued Functions**

   1. A **multivariable vector-valued function** is a function $\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^m$ for some positive integers n and m.

   2. A function $\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^m$ has the form

$$\boldsymbol{F}\left(x_1, x_2, \dots, x_n\right) = \begin{bmatrix} f_1(x_1,x_2,\dots,x_n) \\ f_2(x_1,x_2,\dots,x_n) \\ \vdots \\ f_m(x_1,x_2,\dots,x_n) \end{bmatrix},$$

   where $f_1, f_2, \dots, f_m : \mathbb{R}^n \to \mathbb{R}$ are multivariable real-valued functions. The functions $f_1, f_2, \dots, f_m$ are called the **component functions** of $\boldsymbol{F}$.

   3. A function $\boldsymbol{F} : \mathbb{R}^2 \to \mathbb{R}^3$ has the form

$$\boldsymbol{F}(x, y) = \begin{bmatrix} P(x,y) \\ Q(x,y) \\ R(x,y) \end{bmatrix}.$$

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19. **Derivative of a Multivariable Vector-Valued Function**

   1. Let $\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^m$ be a multivariable vector-valued function. Suppose that $\boldsymbol{F}(x_1, x_2, \dots, x_n)$ is given by the formula

$$\boldsymbol{F}\left(x_1, x_2, \dots, x_n\right) = \begin{bmatrix} f_1(x_1,x_2,\dots,x_n) \\ f_2(x_1,x_2,\dots,x_n) \\ \vdots \\ f_m(x_1,x_2,\dots,x_n) \end{bmatrix}.$$

   The **derivative** (also called the **Jacobian matrix**) of $\boldsymbol{F}$ is the $m \times n$ matrix

$$\boldsymbol{DF}\left(x_1, x_2, \dots, x_n\right) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}.$$

   2. If the point $(x_1, x_2, \dots, x_n)$ is abbreviated by p, then the derivative of $\boldsymbol{F}$ is also denoted $\boldsymbol{DF}(p)$; it is also written $\boldsymbol{DF}_p$ or $\boldsymbol{F}'(p)$. When only the name of the derivative is needed, without listing the variables, it is written $\boldsymbol{DF}$.

   3. If $\boldsymbol{F}(x, y) = \begin{bmatrix} P(x,y) \\ Q(x,y) \\ R(x,y) \end{bmatrix}$, then

$$\boldsymbol{DF}(x, y) = \begin{bmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \\ \frac{\partial R}{\partial x} & \frac{\partial R}{\partial y} \end{bmatrix}.$$

20. **Basic Rules for Derivatives**

Let $\boldsymbol{F}, \boldsymbol{G} : \mathbb{R}^n \to \mathbb{R}^m$ be functions, let c be a real number, and let p be a point in $\mathbb{R}^n$. Suppose that $\boldsymbol{F}$ and $\boldsymbol{G}$ are differentiable. Then

1. $D(\boldsymbol{F} + \boldsymbol{G})(p) = D\boldsymbol{F}(p) + D\boldsymbol{G}(p)$;

2. $D(\boldsymbol{F} - \boldsymbol{G})(p) = D\boldsymbol{F}(p) - D\boldsymbol{G}(p)$;

3. $D(c\boldsymbol{F})(p) = c\,D\boldsymbol{F}(p)$.

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21. **The Jacobian Determinant**

1. Let $\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^n$ be a function. The determinant of the derivative of $\boldsymbol{F}$ is called the **Jacobian determinant** (or just the **Jacobian**) of the function, and is denoted

$$\det D\boldsymbol{F}\left(x_1, x_2, \ldots, x_n\right),$$

or similarly if a different notation for the derivative is used.

2. If the point $(x_1, x_2, \ldots, x_n)$ is abbreviated by p, then the Jacobian determinant of $\boldsymbol{F}$ is also denoted $\det D\boldsymbol{F}(p)$. When only the name of the Jacobian determinant is needed, without listing the variables, it is written $\det D\boldsymbol{F}$.

3. If $\boldsymbol{F}(u, v) = \begin{bmatrix} P(u,v) \\ Q(u,v) \end{bmatrix}$, then the Jacobian determinant of $\boldsymbol{F}$ is sometimes denoted $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ or $\frac{\partial(x,y)}{\partial(u,v)}$.

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22. **Composition of Functions**

Let $\boldsymbol{G} : \mathbb{R}^n \to \mathbb{R}^k$ and $\boldsymbol{F} : \mathbb{R}^k \to \mathbb{R}^m$ be functions. The **composition** of $\boldsymbol{F}$ and $\boldsymbol{G}$ is the function $\boldsymbol{F} \circ \boldsymbol{G} : \mathbb{R}^n \to \mathbb{R}^m$ given by the formula

$$(\boldsymbol{F} \circ \boldsymbol{G})(p) = \boldsymbol{F}(\boldsymbol{G}(p)).$$

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23. **The Chain Rule via Matrix Multiplication**

Let $\boldsymbol{G} : \mathbb{R}^n \to \mathbb{R}^k$ and $\boldsymbol{F} : \mathbb{R}^k \to \mathbb{R}^m$ be functions, and let p be a point in $\mathbb{R}^n$. Suppose that $\boldsymbol{F}$ and $\boldsymbol{G}$ are differentiable. Then

$$D(\boldsymbol{F} \circ \boldsymbol{G})(p) = D\boldsymbol{F}(\boldsymbol{G}(p))\, D\boldsymbol{G}(p),$$

where the multiplication is matrix multiplication.

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24. **The Chain Rule Without Matrices**

Let $z = f(x_1, \ldots, x_n)$, and where each of $x_1, \ldots, x_n$ is a function of $t_1, \ldots, t_m$. Then

$$\frac{\partial z}{\partial t_i} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial z}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial t_i},$$

for each $i = 1, 2, \ldots, m$.

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25. **The Gradient**

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function, and let $\boldsymbol{p} = (x_1, x_2, \ldots, x_n)$ be a point in $\mathbb{R}^n$.

1. The **gradient** of f at $\boldsymbol{p}$, denoted $\nabla f(\boldsymbol{p})$, or $\nabla f\left(x_1, x_2, \ldots, x_n\right)$, or grad $f(\boldsymbol{p})$, or grad $f\left(x_1, x_2, \ldots, x_n\right)$, is defined by

$$\nabla f\left(x_1, x_2, \ldots, x_n\right) = Df\left(x_1, x_2, \ldots, x_n\right)^T = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}^T = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}.$$

2. Whereas the original function f was a multivariable real-valued function $f : \mathbb{R}^n \to \mathbb{R}$, the gradient of f is a multivariable vector-valued function $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$.

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26. **The Gradient and the Chain Rule**

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $\boldsymbol{r} : \mathbb{R} \to \mathbb{R}^n$ be differentiable functions. Then

$$[f(\boldsymbol{r}(t))]' = \nabla f(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t).$$

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27. **The Gradient, Level Curves and Level Surfaces**

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function, let k be a real number, and let (a, b) be a point on the curve given by the equation $f(x, y) = k$. Then $\nabla f(a, b)$ is orthogonal to the tangent line of $f(x, y) = k$ at (a, b).

2. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function, let k be a real number, and let (a, b, c) be a point on the surface given by the equation $f(x, y, z) = k$. Then $\nabla f(a, b, c)$ is orthogonal to the tangent plane of $f(x, y, z) = k$ at (a, b, c).

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28. **Tangent Planes for Implicitly Defined Surfaces**

Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a function, and let $k \in \mathbb{R}$. Let (a, b, c) be a point on the surface defined by the equation $F(x, y, z) = k$. Suppose that the partial derivatives of F exist at (a, b, c). The **tangent plane** to the surface at (a, b, c) has normal vector $\nabla F\,(a, b, c)$, has vector equation

$$\nabla F(a, b, c) \cdot \begin{bmatrix} x - a \\ y - b \\ z - c \end{bmatrix} = 0$$

and has scalar equation

$$F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0.$$

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29. **Tangent Planes For Explicitly Defined Surfaces**

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. Let (a, b) be a point in $\mathbb{R}^2$. Suppose that the partial derivatives of f exist at (a, b). The **tangent plane** to the surface $z = f(x, y)$ at the point (a, b) is given by the equation

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

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30. **Normal Lines for Implicitly Defined Surfaces**

Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a function, and let $k \in \mathbb{R}$. Let (a, b, c) be a point on the surface defined by the equation $F(x, y, z) = k$. The **normal line** to the surface at (a, b, c) has direction vector $\nabla F\,(a, b, c)$, has vector equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} + t \begin{bmatrix} F_x(a, b, c) \\ F_y(a, b, c) \\ F_z(a, b, c) \end{bmatrix},$$

has parametric equations

$$\begin{aligned} x &= a + tF_x(a, b, c) \\ y &= b + tF_y(a, b, c) \\ z &= c + tF_z(a, b, c), \end{aligned}$$

and has symmetric equations

$$\frac{x - a}{F_x(a, b, c)} = \frac{y - b}{F_y(a, b, c)} = \frac{z - c}{F_z(a, b, c)}.$$

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31. **Directional Derivative**

1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function, let $\boldsymbol{p}$ be a vector in $\mathbb{R}^n$ and let $\boldsymbol{u}$ be a unit vector in $\mathbb{R}^n$. The **directional derivative** of f at $\boldsymbol{p}$ in the direction of $\boldsymbol{u}$, denoted $D_{\boldsymbol{u}} f(\boldsymbol{p})$, is

$$D_{\boldsymbol{u}} f(\boldsymbol{p}) = \lim_{h \to 0} \frac{f(\boldsymbol{p} + h\boldsymbol{u}) - f(\boldsymbol{p})}{h},$$

which is defined at all points for which the limit exists.

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. Let $\boldsymbol{p} = (x, y)$, and let $\boldsymbol{u} = (a, b)$ be a unit vector in $\mathbb{R}^2$. Then the directional derivative of f at $\boldsymbol{p}$ in the direction of $\boldsymbol{u}$ is

$$D_{\boldsymbol{u}} f(x, y) = \lim_{h \to 0} \frac{f(x + ha, y + hb) - f(x, y)}{h},$$

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32. **Directional Derivative and the Gradient**

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function, let $\boldsymbol{p}$ be a vector in $\mathbb{R}^n$ and let $\boldsymbol{u}$ be a unit vector in $\mathbb{R}^n$. If the partial derivatives of f at $\boldsymbol{p}$ exist, then $D_{\boldsymbol{u}} f(\boldsymbol{p})$ exists, and

$$D_{\boldsymbol{u}} f(\boldsymbol{p}) = \nabla f(\boldsymbol{p}) \cdot \boldsymbol{u}.$$

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33. **Directional Derivative: Maximal**

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function and let $\boldsymbol{p}$ be a vector in $\mathbb{R}^n$.

1. The unit vector $\boldsymbol{u}$ such that $D_{\boldsymbol{u}} f(\boldsymbol{p})$ is maximal is the unit vector that has the same direction as $\nabla f(\boldsymbol{p})$.

2. The maximal value of the directional derivatives at $\boldsymbol{p}$ is $|\nabla f(\boldsymbol{p})|$.

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34. **Lagrange Multipliers**

Let $f, g, h: \mathbb{R}^3 \to \mathbb{R}$ be functions, and let $k, c \in \mathbb{R}$.

1. Suppose that $f(x, y, z)$, when subject to the constraint $g(x, y, z) = k$, has a global maximum and/or a global minimum. Suppose that $\nabla g(x, y, z) \neq 0$ for any (x, y, z) that satisfies the constraint. To find the global extrema of $f(x, y, z)$ subject to the constraint, first find all values of x, y, z and λ that satisfy

$$\begin{aligned} \nabla f(x, y, z) &= \lambda \nabla g(x, y, z) \\ g(x, y, z) &= k, \end{aligned}$$

then find the value of $f(x, y, z)$ at each of the solutions x, y, z and λ, and find the largest and smallest of these values of $f(x, y, z)$.

2. Suppose that $f(x, y, z)$, when subject to the constraints $g(x, y, z) = k$ and $h(x, y, z) = c$, has a global maximum and/or a global minimum. Suppose that $\nabla g(x, y, z)$ and $\nabla h(x, y, z)$ are non-zero and not parallel for any (x, y, z) that satisfies the constraints. To find the global extrema of $f(x, y, z)$ subject to the constraint, first find all values of x, y, z, λ and μ that satisfy

$$\begin{aligned} \nabla f(x, y, z) &= \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z) \\ g(x, y, z) &= k \\ h(x, y, z) &= c, \end{aligned}$$

then find the value of $f(x, y, z)$ at each of the solutions x, y, z, λ and μ, and find the largest and smallest of these values of $f(x, y, z)$.

Basic Rules for Derivatives

1. $[f(x)+g(x)]' = f'(x)+g'(x)$

2. $[f(x)-g(x)]' = f'(x)-g'(x)$

3. $[cf(x)]' = cf'(x)$

4. $[f(x)g(x)]' = f'(x)g(x)+f(x)g'(x)$

5. $\left[\dfrac{f(x)}{g(x)}\right]' = \dfrac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$

6. $[f(g(x))]' = f'(g(x))g'(x)$

Basic Derivatives

1. $(c)' = 0$

2. $(x)' = 1$

3. $(x^r)' = rx^{r-1}$, for any real number r

4. $(e^x)' = e^x$

5. $(a^x)' = a^x \ln a$

6. $(\ln x)' = \dfrac{1}{x}$

7. $(\ln |x|)' = \dfrac{1}{x}$

8. $(\log_a x)' = \dfrac{1}{\ln a}\dfrac{1}{x}$

9. $(\sin x)' = \cos x$

10. $(\cos x)' = -\sin x$

11. $(\tan x)' = \sec^2 x$

12. $(\sec x)' = \sec x \tan x$

13. $(\csc x)' = -\csc x \cot x$

14. $(\cot x)' = -\csc^2 x$

15. $(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}$

16. $(\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}}$

17. $(\arctan x)' = \dfrac{1}{1+x^2}$

18. $(\text{arcsec}\, x)' = \dfrac{1}{|x|\sqrt{x^2-1}}$

19. $(\text{arccsc}\, x)' = -\dfrac{1}{|x|\sqrt{x^2-1}}$

20. $(\text{arccot}\, x)' = -\dfrac{1}{1+x^2}$

Basic Rules for Indefinite Integrals

1. $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$

2. $\int [f(x) - g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$

3. $\int c f(x)\,dx = c \int f(x)\,dx$

Basic Indefinite Integrals

1. $\int 1\,dx = x + C$

2. $\int x^r\,dx = \dfrac{x^{r+1}}{r+1} + C$ when $r \neq -1$

3. $\int \dfrac{1}{x}\,dx = \ln|x| + C$

4. $\int e^x\,dx = e^x + C$

5. $\int a^x\,dx = \dfrac{a^x}{\ln a} + C$

6. $\int \sin x\,dx = -\cos x + C$

7. $\int \cos x\,dx = \sin x + C$

8. $\int \sec^2 x\,dx = \tan x + C$

9. $\int \sec x \tan x\,dx = \sec x + C$

10. $\int \csc^2 x\,dx = -\cot x + C$

11. $\int \csc x \cot x\,dx = -\csc x + C$

12. $\int \dfrac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$

13. $\int \dfrac{1}{1+x^2}\,dx = \arctan x + C$

14. $\int \dfrac{1}{|x|\sqrt{x^2-1}}\,dx = \operatorname{arcsec} x + C$