Summary of Series Convergence Tests

1. Harmonic Series

1. The **harmonic series** is the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots .$$

2. The harmonic series is divergent.

2. Geometric Series

1. A **geometric series** is any series of the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots ,$$

   where $a, r \in \mathbb{R}$.

2. A geometric series converges to $\dfrac{a}{1-r}$ if $|r| < 1$, and is divergent if $|r| \geq 1$.

3. Divergence Test

Let $\sum_{n=1}^{\infty} a_n$ be a series.

1. If $\lim\limits_{n\to\infty} a_n \neq 0$, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

2. **Caution**: If $\lim\limits_{n\to\infty} a_n = 0$, you CANNOT conclude that the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

4. Integral Test

Let $\sum_{n=1}^{\infty} a_n$ be a series, and let $f : [1, \infty) \to \mathbb{R}$ be function that satisfies the following four properties:

(1) $f(n) = a_n$ for all n.

(2) $f(x)$ is continuous on $[1, \infty)$.

(3) $f(x) > 0$ on $[1, \infty)$.

(4) $f(x)$ is decreasing on $[1, \infty)$.

Then $\sum_{n=1}^{\infty} a_n$ is convergent if and only if $\displaystyle\int_1^{\infty} f(x)\, dx$ is convergent.

5. Remainder Estimate for the Integral Test

Let $\sum_{n=1}^{\infty} a_n$ be a series.

1. Suppose that $\sum_{n=1}^{\infty} a_n$ is convergent by the Integral Test. Let $m \in \mathbb{N}$.

$$\sum_{i=1}^{m} a_i + \int_{m+1}^{\infty} f(x)\,dx \leq \sum_{n=1}^{\infty} a_n \leq \sum_{i=1}^{m} a_i + \int_{m}^{\infty} f(x)\,dx.$$

2. An approximate value for the sum of the series is the average of the upper bound and the lower bound in the above inequalities. The difference between this approximate value and the actual sum of the series is at most half the distance between the upper bound and the lower bound.

6. *p*-Series

1. A ***p*-series** is any series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots,$$

where $p \in \mathbb{R}$.

2. A p-series is convergent if $p > 1$, and is divergent if $p \leq 1$.

7. Comparison Test

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series. Suppose that $a_n \geq 0$ and $b_n \geq 0$ for all $n \in \mathbb{N}$. Suppose that $a_n \leq b_n$ for all $n \in \mathbb{N}$.

1. If $\displaystyle\sum_{n=1}^{\infty} b_n$ is convergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

2. If $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent, then $\displaystyle\sum_{n=1}^{\infty} b_n$ is divergent.

3. **Caution**: If $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent or if $\displaystyle\sum_{n=1}^{\infty} b_n$ is divergent, you CANNOT conclude anything about the other series by the Comparison Test.

8. Limit Comparison Test

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series. Suppose that $a_n \geq 0$ and $b_n \geq 0$ for all $n \in \mathbb{N}$. Suppose that

$$\lim_{n\to\infty} \frac{b_n}{a_n} = L,$$

for some $L \in \mathbb{R}$ or $L = \infty$.

1. Suppose that $0 < L < \infty$. Then either both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent, or both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are divergent.

2. Suppose that $L = 0$. If $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent, then $\displaystyle\sum_{n=1}^{\infty} b_n$ is convergent.

3. Suppose that $L = \infty$. If $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent, then $\displaystyle\sum_{n=1}^{\infty} b_n$ is divergent.

9. Alternating Series Test

Let $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ be an alternating series, where $a_n > 0$ for all $n \in \mathbb{N}$.

1. Suppose that the alternating series satisfies the following two properties:

   (a) the sequence $\{a_n\}_{n=1}^{\infty}$ is decreasing.

   (b) $\lim_{n\to\infty} a_n = 0$.

   Then the alternating series is convergent.

1. The same result holds for alternating series of the form $\sum_{n=1}^{\infty} (-1)^n a_n$.

10. Remainder Estimate for the Alternating Series Test

Let $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ be an alternating series, where $a_n > 0$ for all $n \in \mathbb{N}$. Let $m \in \mathbb{N}$.

1. The m^{th} **remainder** of the alternating series, denoted R_m, is defined by

$$R_m = \sum_{n=1}^{\infty} (-1)^{n-1} a_n - s_m = \sum_{n=m+1}^{\infty} (-1)^n a_n.$$

2. Suppose that the alternating series satisfies the hypotheses of the Alternating Series Test, and hence is convergent. Then $|R_m| \leq a_{m+1}$.

3. The same result holds for alternating series of the form $\sum_{n=1}^{\infty} (-1)^n a_n$.

11. Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series. Suppose that $a_n \neq 0$ for all $n \in \mathbb{N}$. Suppose that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L,$$

for some $L \in \mathbb{R}$ or $L = \infty$.

1. If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

2. If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

3. **Caution**: If $L = 1$, you CANNOT conclude conclude that $\sum_{n=1}^{\infty} a_n$ is either convergent or divergent by the Ratio Test.