A BORDA COUNT FOR PARTIALLY ORDERED BALLOTS

JOHN CULLINAN, SAMUEL K. HSIAO, AND DAVID POLETT

ABSTRACT. The application of the theory of partially ordered sets to voting systems is an important development in the mathematical theory of elections. Many of the results in this area are on the comparative properties between traditional elections with linearly ordered ballots and those with partially ordered ballots. In this paper we present a scoring procedure, called the partial Borda count, that extends the classic Borda count to allow for arbitrary partially ordered preference rankings. We characterize the partial Borda count in the context of weighting procedures and in the context of social choice functions.

1. INTRODUCTION

In this paper we consider the problem of generalizing elections with linearly ordered ballots to those with partially ordered ballots. In principle, partially ordered ballots provide voters with greater flexibility for expressing their true beliefs while still giving the option of submitting a traditional, linearly ordered ballot. The motivation behind introducing partially ordered ballots is simple: it provides a platform that is less restrictive than traditional ranked ballots. For example, given alternatives $a_1,\dots,a_6$, suppose a voter's true preferences are given by Figure 1, representing the fact that a_1 is preferred to all alternatives; a_2

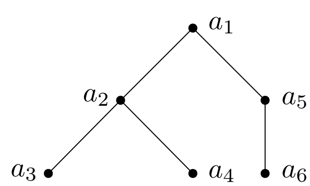


FIGURE 1. Partially ordered ballot

is preferred to a_3 and a_4; a_5 is preferred to a_6; and there are no other preferences among the alternatives. The problem is how to determine winners from the aggregate ballots.

Voting with partially ordered preferences has been an active area of mathematics since Arrow's seminal work (1950). For example, Brown (1974, 1975) explores much of the underlying set theory (filters) associated with partial orders, acyclic relations, and lattice theory applied to voting theory. Ferejohn and Fishburn (1979) introduce the notion of *binary decision rules* and apply it to a generalization of Arrow's theorem; similarly Barthélemy (1982) applies non-linear preference orders to generalizing Arrow's theorem. More recently, Fagin *et al.* (2004) and Ackerman *et al.* (2012) have studied special cases of partially ordered ballots – the so-called *bucket orders* – which are commonly referred to as *weak orders* or *strict weak orders* in the social choice literature.

Our work was originally motivated by the results of Ackerman *et al.* (2012). There, the authors present a method they call *bucket averaging* that applies to the special class of partially ordered preference rankings they call *bucket orders*. The bucket averaging method is an approximation to the *linear extension* method. There, a partially ordered ballot is replaced by all possible linearly ordered (traditional) ballots and the Borda count is applied to these so-called linear extensions. The authors prove that their bucket averaging method gives the same results as the linear extension method, and they further show that the computational complexity of bucket averaging is less expensive than the linear extension method. From a practical perspective, *approval voting* fits naturally into the theory of bucket ordered ballots, where all bucket orders have rank 2.

In this paper we describe a simple score-based voting procedure for partially ordered ballots that is inspired by the classic Borda count for linearly ordered ballots, and yields the classic Borda count results if all voters submit linearly ordered ballots. More generally it yields the same results as the linear extension method (equivalently, Bucket averaging method) if voters submit bucket ordered ballots. We place no restrictions on our partial orders (such as the bucket orders above), and even allow for totally disconnected ballots. We briefly describe our results; detailed definitions appear later in the paper.

Given a partial order on a fixed set A of alternatives, let $\text{down}(a)$, for $a \in A$, be the number of alternatives that are ranked below a, and $\text{incomp}(a)$ be the number of alternatives that are incomparable to a. Assign a weight of $2\,\text{down}(a) + \text{incomp}(a)$ to each $a \in A$. We call this method of assigning weights the partial Borda weighting procedure. This weighting procedure in turn gives rise to a social choice function, which aggregates the weights given to the alternatives according to each voter's preference ranking and declares the alternatives with highest total weight the winners.

Our first main result characterizes partial Borda in the context of weighting procedures; *i.e.*, methods of assigning weights to individual alternatives in an arbitrary partially ordered ballot.

Theorem 1. The partial Borda weighting procedure is the unique weighting procedure, up to affine transformation, that has constant total weights and is linear in the quantities $\text{down}(a)$ and $\text{incomp}(a)$.

Our second main result, which is an adaptation of a result of Young (1974) on the classic Borda count, characterizes partial Borda in the context of social choice functions; *i.e.*, methods of assigning winners to collections of ballots:

Theorem 2. The partial Borda choice function is the unique social choice function (among those whose domain consists of all profiles of partially ordered ballots) that is consistent, faithful, neutral, and has the cancellation property.

In the concluding section of the paper we will discuss further conditions that our voting procedure does and does not satisfy. For instance, partial Borda count satisfies the monotone and Pareto conditions (appropriately generalized to partially ordered ballots) but fails to satisfy plurality. We also show that the partial Borda count specializes to the bucket averaging method of Ackerman *et al.* (2012).

2. Partial Borda weights

In this section we describe an extension of the classic Borda score to the context of partially ordered ballots. We consider elections where the voters submit ballots which consist of partially ordered preference rankings of the alternatives. Such preference rankings can be represented by combinatorial objects called partially ordered sets (*posets*, for short). Throughout this paper, fix a set A of alternatives, with $|A| = n > 1$.

The terminology surrounding the theory of posets differs slightly among social-choice theorists and combinatorialists. We will follow the combinatorial conventions in Stanley's book (1997). Let us recall basic definitions and establish notation needed for the paper.

A relation $\preccurlyeq$ is a *partial order* on A, and $(A, \preccurlyeq)$ is called a *poset*, if the following properties hold:

- Reflexivity: $a \preccurlyeq a$ for all $a \in A$.
- Antisymmetry: If $a \preccurlyeq b$ and $b \preccurlyeq a$ then $a = b$.
- Transitivity: If $a \preccurlyeq b$ and $b \preccurlyeq c$ then $a \preccurlyeq c$.

Write $a \prec b$ if $a \preccurlyeq b$ and $a \neq b$. If $a \prec b$ and there is no $c \in A$ such that $a \prec c \prec b$, then say that b *covers* a. A poset can be visually represented by its *Hasse diagram*, in which elements of the poset are represented by nodes, and a line is drawn from one node a up to another node b whenever b covers a. Two elements a and b in a poset are *comparable* if either $a \preccurlyeq b$ or $b \preccurlyeq a$. They are *incomparable* otherwise. A poset is *linearly ordered* of all pairs of elements are comparable.

Next we go over voting theoretic terminology. Fix an infinite set X, thought of as names of potential voters. Let $\mathbf{R}$ denote the set of real numbers.

- A *profile* is a map p from some finite subset $V \subseteq X$ to the set of partial orders on A. Call V the *voter set of* p. The partial order $p(v)$ is denoted by $\preccurlyeq_v$ when p is understood from the context. We may refer to $\preccurlyeq_v$ as the (partially ordered) *ballot* cast by v.
- A *social choice function* is a map from the set of profiles to the set of non-empty subsets of A.
- A *weight function* is a map from A to $\mathbf{R}$.
- A *weighting procedure* is a map from the set of partial orders on A to the set of weight functions. The weight function associated with a partial order $\preccurlyeq$ is denoted by $w_{\preccurlyeq}$, and $w_{\preccurlyeq}(a)$ is referred to as the *weight* of a.
- A *scoring procedure* is a map from the set of profiles to the set of weight functions. The weight function associated with a profile p is denoted by s_p. We call s_p the *score function* of p, and $s_p(a)$ the *score* of a.

Every weighting procedure naturally gives rise to a scoring procedure, whereby the score function of a profile is defined as the sum of the weight functions of individual ballots. To set notation, if p is a profile with voter set V and $\preccurlyeq_v$ is the partial order associated with $v \in V$, then the corresponding score function of p is defined by

$$s_p(a) = \sum_{v \in V} w_{\preccurlyeq_v}(a). \tag{1}$$

Every scoring procedure in turn gives rise to a social choice function f, where $f(p)$ is the set of alternatives $a \in A$ whose score $s_p(a)$ is highest among all alternatives.

Given a partial order $\preccurlyeq$, define the *down set* and *incomparable set* of $a \in A$ by

$$\begin{aligned} \mathrm{Down}(a) &= \{b \in A \mid b \prec a\}; \\ \mathrm{Incomp}(a) &= \{b \in A \mid b \text{ is incomparable to } a\}. \end{aligned}$$

Let $\mathrm{down}(a) = |\mathrm{Down}(a)|$ and $\mathrm{incomp}(a) = |\mathrm{Incomp}(a)|$. When it is important to emphasize the dependency on $\preccurlyeq$ we write $\mathrm{down}_{\preccurlyeq}(a)$ and $\mathrm{incomp}_{\preccurlyeq}(a)$.

We propose the following weighting procedure.

Definition 1. The *partial Borda weighting procedure* is the weighting procedure that associates a partial order $\preccurlyeq$ with the weight function $w_\preccurlyeq : A \to \mathbf{R}$ given by

$$w_\preccurlyeq(a) = 2\,\mathrm{down}_\preccurlyeq(a) + \mathrm{incomp}_\preccurlyeq(a). \tag{2}$$

The corresponding scoring procedure defined by (1) is called the *partial Borda scoring procedure*, and the score function s_p is called the *partial Borda score*. The corresponding social choice function that chooses alternatives with the highest score is called the *partial Borda choice function.*

Remark 1. We offer another, equivalent, interpretation of the partial Borda weight function that gives some insight into (2) and facilitates later proofs. Given a partial order $\preccurlyeq$ on A, start by giving each $a \in A$ a weight of $n-1$. (We think of a as initially receiving one "point" for each of the other alternatives.) Then, for every pair $a, b \in A$ with $a \prec b$, we decrease the weight of a by 1 and increase the weight of b by 1. Informally, an alternative must "give away" one point to every alternative that is ranked above it. After reallocating weights in this manner, the final weight assignments agree with (2).

Example. Suppose that there are 6 voters and 5 alternatives, $A = \{a, b, c, d, e\}$, and the profile p consisting of the posets submitted by these voters is depicted in Figure 2. Next to each alternative we indicate the weight assigned by the partial Borda weighting procedure. Note that the weights depend only on the poset and not on the voter who submitted it.

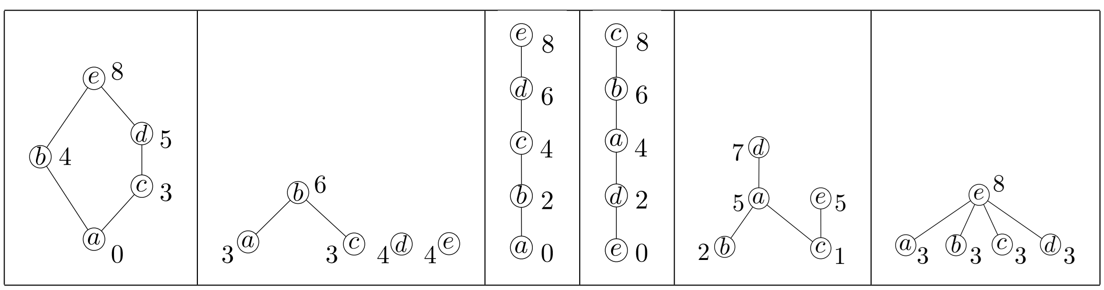


FIGURE 2. Partial Borda weights for several posets

The scores of the alternatives are as follows: $s_p(a) = 15$, $s_p(b) = 23$, $s_p(c) = 22$, $s_p(d) = 27$, $s_p(e) = 33$. Thus, the partial Borda choice set is $\{e\}$.

We will show that the partial Borda weighting procedure is characterized, up to affine transformation, by the following two properties:

- *Constant total weight:* There is a constant $\delta \in \mathbf{R}$ such that $\sum_{a\in A} w_\preccurlyeq(a) = \delta$ for all partial orders $\preccurlyeq$.
- *Linearity:* There are constants $\alpha, \beta, \gamma \in \mathbf{R}$ such that
$$w_\preccurlyeq(a) = \alpha \cdot \mathrm{down}(a) + \beta \cdot \mathrm{incomp}(a) + \gamma$$
for all $a \in A$ and partial orders $\preccurlyeq$.

The constant total weight condition holds for classic Borda (where $\delta = \sum_{i=1}^{n}(i-1) = n(n-1)/2$) and is a reasonable condition to impose on a weighting procedure if we wish to avoid favoring certain preference rankings over others.

The linearity condition implies that, just as in classic Borda, if a voter changes the preference relation among alternatives that are ranked below some alternative a, or among alternatives that are not comparable to a, then the weight assigned to a should not change. (Of course in classic Borda, there would be no incomparable alternatives.) On the other hand, if a voter changes preferences by, for example, taking an alternative originally ranked below a and making it incomparable to a, then the weight assigned to a by the voter changes by a constant amount, in this case $\beta - \alpha$.

Our first main result is that the constant total weight and linearity conditions characterize the partial Borda weighting procedure, up to an affine transformation.

Theorem 1. *The partial Borda weighting procedure $w_{\preccurlyeq}$ satisfies the constant total weight and linearity conditions. Conversely, if $w'_{\preccurlyeq}$ is any weighting procedure satisfying these conditions, with constant total weight δ and linearity coefficients α, β, and γ, then*

$$\alpha = 2\beta = \frac{2\,(\delta - \gamma n)}{n(n-1)} \tag{3}$$

and

$$w'_{\preccurlyeq}(a) = \beta \cdot w_{\preccurlyeq}(a) + \left[\frac{\delta}{n} - \beta(n-1)\right]. \tag{4}$$

Proof. The partial Borda weighting procedure is linear by definition. With the interpretation of the weight function given in Remark 1 it is clear that, because each of the n alternatives initially receives a weight of $n-1$, the sum of the weights of all the alternatives is always $n(n-1)$.

Conversely, suppose we are given a weighting procedure w' with $\delta = \sum_{a\in A} w'_{\preccurlyeq}(a)$ and $w'_{\preccurlyeq}(a) = \alpha \cdot \mathrm{down}(a) + \beta \cdot \mathrm{incomp}(a) + \gamma$ for constants $\alpha, \beta, \gamma, \delta \in \mathbf{R}$. Pick an arbitrary linear ordering $\preccurlyeq_1$ of the elements of A, say $a_1 \prec_1 a_2 \prec_1 \cdots \prec_1 a_n$. Then $\delta = \sum_{i=1}^{n} w'_{\preccurlyeq_1}(a_i) = \sum_{i=1}^{n}(\alpha(i-1) + \beta \cdot 0 + \gamma) = \alpha n(n-1)/2 + \gamma n$. Next, let $\preccurlyeq_2$ be the partial order in which all pairs of alternatives are incomparable. Then $\delta = \sum_{a\in A} w'_{\preccurlyeq}(a) = \beta\, n(n-1) + \gamma n$. Solving for α and β gives (3), and we also get $\gamma = \delta/n - \beta(n-1)$. Moreover, $w'_{\preccurlyeq}(a) = 2\beta \cdot \mathrm{down}(a) + \beta \cdot \mathrm{incomp}(a) + \gamma = \beta \cdot w_{\preccurlyeq}(a) + \gamma$, which proves (4). □

If two weighting procedures w and w' are related by $w'_{\preccurlyeq}(a) = t \cdot w_{\preccurlyeq}(a) + u$ for constants t, u, such that $t > 0$, and if the corresponding score functions s and s' are defined as in (1), then clearly $s_p(a) > s_p(b)$ if and only if $s'_p(a) > s'_p(b)$ for all $a, b \in A$ and all profiles p. Thus the corresponding social choice functions defined by these two scoring procedures are equal. Accordingly, we have the following consequence of Theorem 1.

Corollary 1. *If a weighting procedure has constant total weights and is linear with $\beta > 0$, then the corresponding social choice function (defined in the usual manner by aggregating weights and declaring alternatives with the highest score to be winners) is the partial Borda choice function.*

3. Characterization of the partial Borda choice function

Our main result in this section is a characterization of the partial Borda choice function in terms of four properties:

Theorem 2. *The partial Borda choice function is the unique social choice function that is consistent, faithful, neutral, and has the cancellation property.*

This result is anticipated by Young (1974), who characterizes the classic Borda choice function in terms of these same four properties, one caveat being that we adopt a different notion of faithfulness that seems better suited to posets. Indeed, Young suggests a way to extend the definition of Borda score to partially ordered ballots and mentions that his characterization should extend to partially ordered preference rankings. Our partial Borda score turns out to be closely related to the definition he proposes (see Lemma 2 and the preceding discussion). However, our proof of uniqueness, which is specific to partially ordered preferences, is quite different from Young's proof, which is specific to linearly ordered preferences. Neither result is an obvious specialization of the other. Our proof is elementary and self-contained; in particular we avoid the linear algebraic and graph theoretic techniques used by Young.

We establish notation and some preliminary results, followed by an outline of the proof of Theorem 2 before going through the details. As before, A is a fixed set of alternatives with $|A| = n$. If p_1 and p_2 are disjoint profiles, meaning their underlying voter sets V_1 and V_2 are disjoint, then $p_1 + p_2$ denotes the profile with voter set $V_1 \cup V_2$, which when restricted to V_i agrees with p_i. Let f be a social choice function. For a profile p and $a \neq b \in A$, let $\pi_{ab}(p)$ denote the number of voters who rank a above b. Let us define consistent, faithful, neutral, and the cancellation property for an arbitrary social choice function f based on partially ordered preferences.

- *Consistent:* For disjoint profiles p_1 and p_2, if $f(p_1) \cap f(p_2) \neq \emptyset$ then $f(p_1) \cap f(p_2) = f(p_1 + p_2)$. (Consistency is also called the *convexity* criterion when the underlying geometry is emphasized (see Woodall (1994).)
- *Faithful:* For any profile p consisting of just one voter, if $a \in A$ is an alternative and the voter ranks b above a for some $b \in A$, then $a \notin f(p)$.
- *Neutral:* For any profile p and permutation σ of A, $f(\sigma(p)) = \sigma(f(p))$. Here $\sigma(p)$ denotes the profile in which every voter relabels the alternatives according to σ; that is, a voter prefers a over b in p if and only if that voter prefers $\sigma(a)$ to $\sigma(b)$ in $\sigma(p)$.
- *Cancellation property:* For any profile p, if $\pi_{ab}(p) = \pi_{ba}(p)$ for all $a \neq b \in A$ then $f(p) = A$.

Two profiles p_1 and p_2 with voter sets V_1 and V_2 are *isomorphic* (respectively, *anti-isomorphic*) if there is a bijection $\phi : V_1 \to V_2$ such that for all $v \in V_1$, the partial order in p_1 corresponding to voter v is identical to (respectively, the dual of) the partial order in p_2 corresponding to voter $\phi(v)$. (Two partial orders $\preccurlyeq$ and $\preccurlyeq'$ on A are dual to each other if for all $a, b \in A$, $a \preccurlyeq b$ if and only if $b \preccurlyeq' a$.)

Lemma 1. *Suppose f is consistent and has the cancellation property. If p and q are isomorphic profiles, then $f(p) = f(q)$.*

Proof. First suppose p and q are disjoint. Let t be a profile that is disjoint from p and q and anti-isomorphic to p, hence also q. By the cancellation property, $f(p+t) = A = f(t+q)$.

By consistency, $f(p) = f(p) \cap A = f(p+t+q) = A \cap f(q) = f(q)$. If p and q are not disjoint, create isomorphic copies p' and q' that are disjoint from each other and from p and q. We then have $f(p) = f(p') = f(q') = f(q)$. □

This lemma gives us the flexibility to pass freely between a profile and its isomorphism class. When we refer to p as a profile we will usually mean p is an isomorphism class of profiles. When we need to refer to the underlying voter set, we assume that an arbitrary representative of the isomorphism class has been chosen. The "+" operation that was originally defined on profiles with disjoint voter sets extends naturally to a well-defined commutative operation on isomorphism classes.

If $a \neq b \in A$, let $\langle a,b\rangle$ denote a profile with just a single voter, who ranks b above a and expresses no other preferences. The profile $\langle a,b\rangle + \langle b,a\rangle$ is called a 2-*cycle.* We say a profile p is *reduced* if $p = \sum_{i=1}^{k} \langle a_i, b_i\rangle$ for some $a_1, b_1, \dots, a_k, b_k \in A$ such that $a_i \neq b_j$ for all i, j. Given profiles p and q and score functions s_p and s'_q, we say that s_p is a *shift* of s'_q if there is a constant N (possibly depending on p and q) such that $s_p(a) = s'_q(a) + N$ for all $a \in A$; in this situation it is clear that the corresponding choice sets $f(p)$ and $f'(q)$ are equal.

Young (1974) uses the score function $s'_p(a) = \sum_{b\in A-\{a\}}(\pi_{ab}(p) - \pi_{ba}(p))$ as the working definition of the classic Borda score function (where p is a profile of linearly ordered ballots). This score function is easily seen to produce the same choice set as the usual Borda score function in which the ith lowest ranked alternative in a ballot is assigned weight $i-1$. Young proposes using s'_p to define a Borda score for partial orders. We show that our partial Borda score s_p is a shift of the one proposed by Young:

Lemma 2. *For a profile p with m voters, the partial Borda score of an alternative $a \in A$ is*

$$s_p(a) = m\cdot(n-1) + \sum_{b\in A-\{a\}} (\pi_{ab}(p) - \pi_{ba}(p)) \tag{5}$$

Proof. With the interpretation of the partial Borda weight function given in Remark 1, initially each of the m voters assigns weight $n-1$ to each alternative $a \in A$, so a receives an initial total score of $m(n-1)$. But then a receives an additional $\sum_{b\in A-\{a\}} \pi_{ab}(p)$ points from the lower-ranked alternatives while giving away $\sum_{b\in A-\{a\}} \pi_{ba}(p)$ points to higher-ranked alternatives. The resulting net score is exactly as in (5). □

Let us give a quick overview of the proof of Theorem 2. It is immediate from the definitions that the partial Borda choice function satisfies consistency and neutrality. Partial Borda satisfies faithfulness by Proposition 3, and the cancellation property by Equation (5). Now we outline the proof of uniqueness. Given a social choice function f satisfying these four properties and a profile p, we will construct a new profile q that is reduced, and has the property that $f(p) = f(q)$ and the partial Borda scores s_p and s_q are shifts of each other. Lemmas 3, 4, and 5 will show that each step of the construction preserves $f(p)$ as well as s_p, up to a shift. The steps to constructing q are: (1) add together profiles of the form $\langle a,b\rangle$, one for each instance a voter in p expresses preference for b over a (see Lemma 3); (2) remove 2-cycles from q (see Lemma 4); (3) replace terms of the form $\langle a,b\rangle + \langle b,c\rangle$ with $\langle a,c\rangle$ (see Lemma 5); (4) repeat steps (2) and (3) until q is reduced. Lemma 6 is a specialized result needed for Lemma 7, which in turn gives a simple expression for $f(q)$. Having q be reduced will be an important assumption in deriving this expression. It will then be clear that $f(q)$ coincides with the partial Borda choice set of q. Because q maintains the same choice set and partial Borda choice set as p, we conclude $f(p)$ is the partial Borda choice set of p.

For the remaining discussion, assume f is a social choice function that is neutral, consistent, faithful, and has the cancellation property, and s_p is the partial Borda score. We will frequently refer to the following property, which is an immediate consequence of consistency:

Deletion Property: If p and q are profiles and $f(p) = A$, then $f(p+q) = f(q)$.

Lemmas 3 to 8 constitute our proof of the "uniqueness" part of Theorem 2.

Lemma 3. *For any profile p, there is a profile $q = \sum_{i=1}^{k} \langle a_i, b_i \rangle$ such that $f(q) = f(p)$ and s_q is a shift of s_p.*

Proof. Let V be the voter set for p and

$$q = \sum_{v \in V} \sum_{\substack{a,b \in A \\ a \prec_v b}} \langle a, b \rangle \quad \text{and} \quad q' = \sum_{v \in V} \sum_{\substack{a,b \in A \\ a \prec_v b}} \langle b, a \rangle$$

By the cancellation property, $f(q'+q) = f(p+q') = A$. Applying the deletion property twice, we get $f(p) = f(p+q'+q) = f(q)$. As for Borda scores, because $\pi_{ab}(p) = \pi_{ab}(q)$ for all a, b, then according to (5), s_q must be a shift of s_p. □

Lemma 4. *Suppose q is obtained by removing a 2-cycle $\langle a, b \rangle + \langle b, a \rangle$ from a profile p. Then $f(q) = f(p)$ and s_q is a shift of s_p.*

Proof. By the deletion property, $f(p) = f(q + \langle a, b \rangle + \langle b, a \rangle) = f(q)$. Removing a 2-cycle from p does not change $\pi_{xy}(p) - \pi_{yx}(p)$ for any $x \neq y \in A$. Hence, by (5), s_q is a shift of s_p. □

In the previous lemma we allow $p = \langle a, b \rangle + \langle b, a \rangle$, in which case q is the "empty profile" that has no voters. In this situation we set $f(q) = A$ and $s_q(a) = 0$ for all $a \in A$.

Lemma 5. *Suppose q is obtained by replacing a copy of $\langle a, b \rangle + \langle b, c \rangle$ in a profile p by $\langle a, c \rangle$, where a, b, c are distinct. Then $f(q) = f(p)$ and s_q is a shift of s_p.*

Proof. Let $t = \langle a, b \rangle + \langle b, c \rangle + \langle c, a \rangle$. We first show that $f(t) = A$. Let $t' = \langle b, a \rangle + \langle c, b \rangle + \langle a, c \rangle$. Suppose one of the elements from the set $\{a, b, c\}$, let us say a, is in $f(t)$. Let σ be the cyclic permutation (abc) of A. Then $\sigma(p) = p$. By neutrality, $b = \sigma(a) \in \sigma(f(t)) = f(\sigma(t)) = f(t)$. Applying σ again, we get $c \in f(t)$. Let τ be the transposition (ac). Then $\tau(t)$ is isomorphic to t', and hence $f(t') = f(\tau(t)) = \tau(f(t)) = f(t)$. In particular, $f(t) \cap f(t') \neq \emptyset$. Applying the cancellation property and then consistency, $A = f(t+t') = f(t) \cap f(t')$. Therefore, $f(t) = A$. Suppose instead that some alternative $d \notin \{a, b, c\}$ is in $f(t)$. Then $d = \tau(d) \in \tau(f(t)) = f(t')$, which means $f(t) \cap f(t') \neq \emptyset$. As before it follows that $f(t) = A$.

To show $f(q) = f(p)$, we write $p = \langle a, b \rangle + \langle b, c \rangle + p'$ for some p'. Then apply the deletion property twice, to get

$$f(p) = f((\langle a, c \rangle + \langle c, a \rangle) + (\langle a, b \rangle + \langle b, c \rangle + p')) = f(t + \langle a, c \rangle + p') = f(\langle a, c \rangle + p') = f(q).$$

Finally, it is easy to verify that for any $x \in A$,

$$\sum_{y \in A - \{x\}} (\pi_{xy}(q) - \pi_{yx}(q)) = \sum_{y \in A - \{x\}} (\pi_{xy}(p) - \pi_{yx}(p)).$$

Therefore, by (5), s_q is a shift of s_p. □

Lemma 6. *Let $a_1, b_1, a_2, b_2, \dots, a_k, b_k$ be elements of A such that $b_1, \dots, b_k$ are distinct, $a_1, \dots, a_k$ are not necessarily distinct, and $a_i \neq b_j$ for all i, j. Then $f(\sum_{i=1}^k \langle a_i, b_i \rangle) = \{b_1, \dots, b_k\}$.*

Proof. Let $c_1, \dots, c_\ell$ be distinct elements such that, as sets, $\{c_1, \dots, c_\ell\} = \{a_1, \dots, a_k\}$.

First consider the case $\ell = k = 1$. Suppose $c \in f(\langle c_1, b_1 \rangle)$ for some c distinct from c_1 and b_1. Let τ be the permutation of A that transposes c_1 and b_1. By neutrality, $c = \tau(c) \in \tau(f(\langle c_1, b_1 \rangle)) = f(\langle b_1, c_1 \rangle)$. Thus $c \in f(\langle c_1, b_1 \rangle) \cap f(\langle b_1, c_1 \rangle)$. Then by the cancellation property and consistency, $A = f(\langle c_1, b_1 \rangle + \langle b_1, c_1 \rangle) = f(\langle c_1, b_1 \rangle) \cap f(\langle b_1, c_1 \rangle)$. Hence $f(\langle c_1, b_1 \rangle) = A$, contradicting that fact that, by faithfulness, $c_1 \notin f(\langle c_1, b_1 \rangle)$. We conclude that $f(\langle c_1, b_1 \rangle)$ cannot contain alternatives other than c_1 or b_1. Having just observed that it cannot contain c_1, we must have $f(\langle c_1, b_1 \rangle) = \{b_1\}$.

Continue to assume $\ell = 1$ and proceed by induction on k. Let $k \geq 2$, $\{b_1, \dots, b_k\}$ be a subset of A, and $c_1 \in A$ be outside of this subset. By consistency, $f(\sum_{i=1}^k \langle b_i, c_1 \rangle) = \{c_1\}$. This means c_1 cannot be in $f(\sum_{i=1}^k \langle c_1, b_i \rangle)$ because otherwise we would have, by consistency and cancellation, $\{c_1\} = f(\sum_{i=1}^k \langle b_i, c_1 \rangle) \cap f(\sum_{i=1}^k \langle c_1, b_i \rangle) = f(\sum_{i=1}^k (\langle b_i, c_1 \rangle + \langle c_1, b_i \rangle)) = A$, a contradiction.

Suppose next that $d \in f(\sum_{i=1}^k \langle c_1, b_i \rangle)$ for some d distinct from c_1, b_1, $\dots$, b_k. With τ as before, $d = \tau(d) \in f(\tau(\sum_{i=1}^k \langle c_1, b_i \rangle)) = f(\langle b_1, c_1 \rangle + \sum_{i=2}^k \langle c_1, b_i \rangle)$. Therefore, $d \in f(\sum_{i=1}^k \langle c_1, b_i \rangle) \cap f(\langle b_1, c_1 \rangle + \sum_{i=2}^k \langle c_1, b_i \rangle) = f(\langle c_1, b_1 \rangle + \langle b_1, c_1 \rangle + \sum_{i=2}^k \langle c_1, b_i \rangle + \sum_{i=2}^k \langle c_1, b_i \rangle) = A \cap f(\sum_{i=2}^k \langle c_1, b_i \rangle) \cap f(\sum_{i=2}^k \langle c_1, b_i \rangle) = \{b_2, \dots, b_k\}$, by induction. It follows that $d \in \{b_2, \dots, b_k\}$, a contradiction. We have now shown that $f(\sum_{i=1}^k \langle c_1, b_i \rangle)$ cannot contain any alternatives outside the set $\{b_1, \dots, b_k\}$. There must be some b_j in the set $f(\sum_{i=1}^k \langle c_1, b_i \rangle)$. By transposing this b_j with each of the remaining b_i's and using neutrality, we conclude that every b_i is in this set. This completes the proof of the lemma in the case $\ell = 1$ and k arbitrary.

We now consider the general case $1 \leq \ell \leq k$. Consider the cyclic permutation $\sigma = (c_1 c_2 \cdots c_\ell)$ of A. Let $p = \sum_{i=1}^k \langle a_i, b_i \rangle$ and $q = p + \sigma(p) + \sigma^2(p) + \cdots + \sigma^{\ell-1}(p)$. Then $f(q) = f(\sum_{i=1}^\ell \sum_{j=1}^k \langle c_i, b_j \rangle) = \bigcap_{i=1}^\ell f(\sum_{j=1}^k \langle c_i, b_j \rangle) = \bigcap_{i=1}^k \{b_1, \dots, b_k\} = \{b_1, \dots, b_k\}$, by the $\ell = 1$ case.

Suppose there is some $d \notin \{a_1, b_1, \dots, a_k, b_k\}$ such that $d \in f(p)$. By neutrality $d = \sigma^i(d) \in f(\sigma^i(p))$ for $i = 1, \dots, \ell - 1$. Therefore $d \in f(p) \cap f(\sigma(p)) \cap \cdots \cap f(\sigma^{\ell-1}(p)) = f(q) = \{b_1, \dots, b_k\}$, a contradiction. No such d can exist. It follows that $f(p) \subseteq \{a_1, \dots, a_k\} \cup \{b_1, \dots, b_k\}$.

Suppose next that there is some $a \in \{a_1, \dots, a_k\}$ such that $a \in f(p)$. Let $S = \{i : a_i = a\}$. Then $a \in \{a\} \cap f(p) = f(\sum_{i \in S} \langle b_i, a \rangle) \cap f(\sum_{i=1}^k \langle a_i, b_i \rangle) = f(\sum_{i \in S} (\langle b_i, a_i \rangle + \langle a_i, b_i \rangle) + \sum_{i \notin S} \langle a_i, b_i \rangle) = f(\sum_{i \notin S} \langle a_i, b_i \rangle)$. As in the previous paragraph one can show that $f(\sum_{i \notin S} \langle a_i, b_i \rangle)$ is a subset of $\{a_i : i \notin S\} \cup \{b_i : i \notin S\}$. We have reached a contradiction as a is not contained in the latter set. Therefore no such a exists. It follows that $f(p) \subseteq \{b_1, \dots, b_k\}$ and hence the permutation σ fixes $f(p)$. We have $f(p) = f(p) \cap \sigma(f(p)) \cap \cdots \cap \sigma^{\ell-1}(f(p)) = f(q) = \{b_1, \dots, b_k\}$. □

Given a reduced profile $p = \sum_{i=1}^k \langle a_i, b_i \rangle$, let $\lambda(b)$, $b \in A$, be the number of voters who express a preference for b; that is, $\lambda(b) = |\{i : b = b_i\}|$. Also let $\mu = \max\{\lambda(b) : b \in A\}$.

Lemma 7. *If p is a reduced profile as before, then $f(p) = \{b \in A : \lambda(b) = \mu\}$.*

Proof. Order the alternatives $b_1, \dots, b_n$ so that $\mu = \lambda(b_1) \geq \lambda(b_2) \geq \cdots \geq \lambda(b_\ell) > \lambda(b_{\ell+1}) = \cdots = \lambda(b_n) = 0$. Thus for each $i = 1, \dots, \ell$, there are $\lambda(b_i)$ voters who cast ballots of the form $\langle *, b_i \rangle$, where $*$ stands for some alternative whose name will not matter. Furthermore, none of the voters show any preference for any of the alternatives $b_{\ell+1},\dots,b_n$. The summands of p can be arranged as follows in a left-justified array of ℓ rows, with the ith row having $\lambda(b_i)$ terms:

$$\begin{array}{lr} p = \langle *, b_1\rangle + \langle *, b_1\rangle + \cdots + \langle *, b_1\rangle & (\lambda(b_1) \text{ terms}) \\ \quad + \langle *, b_2\rangle + \langle *, b_2\rangle + \cdots \;\vdots & (\lambda(b_2) \text{ terms}) \\ \quad \vdots & \vdots \\ \quad + \langle *, b_\ell\rangle + \cdots & (\lambda(b_\ell) \text{ terms}) \end{array}$$

Let p_i be the sum of the terms in the ith column, for $i = 1,\dots,\lambda(b_1)$. Let $m = |\{i : \lambda(b_i) = \mu\}|$, the number of terms in the last column. Because each p_i is reduced, Lemma 6 is applicable, and we have $\{b_1,\dots,b_\ell\} = f(p_1) \supseteq f(p_2) \supseteq \cdots \supseteq f(p_\mu) = \{b_1,\dots,b_m\}$. By consistency, $f(p) = f(p_1 + \cdots + p_\mu) = f(p_1) \cap \cdots \cap f(p_\mu) = \{b_1,\dots,b_m\} = \{b \in A : \lambda(b) = \mu\}$. □

Lemma 8. *f is the partial Borda choice function.*

Proof. Given any profile p, let q be as in Lemma 3. Repeatedly apply Lemmas 4 and 5, removing 2-cycles from q and replacing terms of the form $\langle a,b\rangle + \langle b,c\rangle$ with $\langle a,c\rangle$. Each application of one of these lemmas will decrease the number of terms in q, so the procedure will eventually terminate, resulting in a reduced profile q such that $f(p) = f(q)$ and s_q is a shift of s_p. The choice set $f(q)$, as described in Lemma 7, clearly agrees with partial Borda choice set for q. Furthermore, since s_q is a shift of s_p, the partial Borda choice set of q equals that of p. □

4. FURTHER PROPERTIES OF PARTIAL BORDA

We explore connections between the partial Borda count and certain voting systems involving bucket ordered ballots. We also consider some well-known properties in the mathematical theory of elections, namely the monotone and Pareto conditions, in the context of the partial Borda count.

In Ackerman *et al.* (2012), the authors study a scoring procedure for profiles consisting of *bucket ordered* ballots. We will show that the partial Borda score function specializes to theirs when we restrict to bucket orders. As before A is a fixed set of n alternatives. Recall that a partial order $\preccurlyeq$ on A is called a *bucket order* (or *bucket poset*) if A can be partitioned into a disjoint union $A = A_1 \cup A_2 \cup \cdots \cup A_k$ of nonempty sets (called *buckets*) such that for all $a,b \in A$, we have $a \prec b$ if and only if $a \in A_i$ and $b \in A_j$ for some $i < j$. We refer to A_i as an *equivalence class*.

One way to determine the social choices from a profile of bucket orders is to replace each bucket order with the (in principle, much larger) set of all its linear extensions and then use the usual Borda function to score the linearly ordered ballots. One of the main results of Ackerman *et al.* is to show that the result of such a scoring procedure is the same as their *bucket averaging* method, defined as follows. Given a bucket order, which partitions A into equivalence classes $A_1, \dots, A_k$, let $n_i = |A_i|$ and assign the following weight to each member of A_i:

$$\frac{(n_1+\cdots+n_{i-1}) + (n_1+\cdots+n_{i-1}+1) + \cdots + (n_1+\cdots+n_{i-1}+n_i-1)}{n_i} = \frac{2(n_1+\cdots n_{i-1}) + n_i - 1}{2}.$$

In other words, the weight assigned to each alternative in A_i is the average of the Borda weights of these alternatives in some (any) linear extension of the bucket order. We can immediately deduce the following result describing how partial Borda weight function extends the bucket averaging weight function.

Proposition 1. *Suppose $\preccurlyeq$ is a bucket order on A. Then the bucket averaging method of Ackerman et al. assigns a weight of $w_{\preccurlyeq}(a)/2$ to each $a \in A$, where $w_{\preccurlyeq}(a)$ is the partial Borda weight of a. Consequently, the partial Borda count, when restricted to bucket orders, will produce the same social choices as the bucket averaging method.*

Bucket orders arise naturally in Borda count elections that allow for truncated ballots. In particular, suppose we modify a traditional Borda count so that voters are allowed to rank a proper subset and give the unranked alternatives a score of zero. An extreme version, so-called *bullet voting*, is when a voter only ranks a single alternative (Niemi 1984). For example (using our convention of multiplying the Borda scores by 2), given five alternatives $a_1,\dots,a_5$, a voter could give scores of 8 to a_1, 6 to a_2, and 0 to a_3, a_4, a_5. A bullet vote in this example would give a single alternative a score of 8, and all others a score of 0.

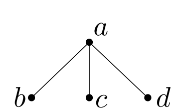
One can create a bucket order from a truncated linear order by placing all unranked alternatives into a single equivalence class at the bottom. However, for such a partial ordering of the alternatives, our partial Borda procedure would give each of the unranked alternatives a weight equal to the size of the equivalence class minus one, whereas the truncated Borda procedure would given them a weight of zero.

Proposition 2. *The truncated Borda procedure and the partial Borda procedure do not necessarily produce the same social choices.*

Proof. Suppose the following ballots are submitted in a truncated Borda count election, with the horizontal lines indicating the truncation:

$$\begin{bmatrix}\underline{a}\\ b\\ c\\ d\end{bmatrix}\times 5 \qquad \begin{bmatrix}b\\ c\\ d\\ a\end{bmatrix}\times 4.$$

Then a receives a score of 30, while b, c, and d receive scores of 24, 16, and 8, respectively. However, if we replace the five truncated ballots with



then the partial Borda scores of a, b, c, and d are 30, 34, 26, and 18 points, respectively, and b is the social choice. □

Next we consider two properties related to monotonicity. Recall the following definitions from Saari (1995). A social choice function based on linearly ordered preferences is said to satisfy the *monotone condition* if, when $a \in A$ is a social choice from a given profile p, and the only voters to change their preferences give a a higher ranking (but preserving the original relations between other alternatives), then a is a social choice in the new profile. The function satisfies the *Pareto condition* if, whenever every voter in p prefers a over b then b is not a social choice.

The monotone and Pareto conditions can be defined as follows for an arbitrary social choice function f, whose domain is the set of profiles whose ballots are partial orders:

- *Monotone condition:* Let p be a profile and a be in the choice set $f(p)$. Suppose one of the voters changes his original preference order from $\preccurlyeq$ to a preference order $\preccurlyeq'$ with the property that for all $b, c \in A - \{a\}$,

$$b \prec c \Longleftrightarrow b \prec' c, \quad b \prec a \Longrightarrow b \prec' a, \quad \text{and} \quad a \not\prec b \implies a \not\prec' b$$

  Then a is in the choice set $f(p')$ of the new profile p'.

- *Pareto condition:* For a profile p and alternatives $a, b \in A$, if every voter in p prefers b over a, then $a \notin f(p)$.

Proposition 3. *The partial Borda choice function satisfies the monotone and Pareto conditions.*

Proof. Let $\preccurlyeq$ be a partial order on A. We claim that the partial Borda weight function $w_{\preccurlyeq}$ is strictly order-preserving; i.e., $a \prec b$ implies $w_{\preccurlyeq}(a) < w_{\preccurlyeq}(b)$. Suppose $a \prec b$. With the interpretation of partial Borda weights given in Remark 1, b and a initially have the same weight $(n-1)$. For every point that a gains from a lower ranked alternative, b also gains a point, and for every point b loses to a higher ranked alternative, a also loses a point. Furthermore, a loses an extra point to b. Thus in the end $w_{\preccurlyeq}(b) \geq w_{\preccurlyeq}(a) + 2 > w_{\preccurlyeq}(a)$. Having shown that $w_{\preccurlyeq}$ strictly order-preserving, it follows that if every voter in a profile p prefers b over a, then $s_p(b) > s_p(a)$. Hence the Pareto condition is satisfied.

Next, suppose $a \in f(p)$ and that one voter changes his preference order, as described in the definition of the monotone condition above, resulting in a new profile p'. Again using Remark 1, it is clear that $w_{\preccurlyeq'}(a) \geq w_{\preccurlyeq}(a)$ and that $w_{\preccurlyeq'}(b) \leq w_{\preccurlyeq}(b)$ for all $b \neq a$. The implies $s_{p'}(a) \geq s_p(a)$ and $s_{p'}(b) \leq s_p(b)$, and consequently $a \in f(p')$. Hence the monotone condition is satisfied. □

Lastly, we show that partial Borda, unlike classic Borda, does not satisfy the *plurality condition*: if the number of ballots in which a is the single most preferred alternative is greater than the number of ballots in which alternative b is shown any preference over another alternative, then a receives a higher score than b.

Proposition 4. *Partial Borda count does not satisfy the plurality condition.*

Proof. To see this, suppose there are 19 voters and three alternatives: a_1, a_2, and a_3. Partition the voters into two subsets of size 10 and 9 with ballots as in Figure 3, respectively. The partial Borda scores for this profile are $s(a_1) = 40$, $s(a_2) = 28$, and $s(a_3) = 46$. □

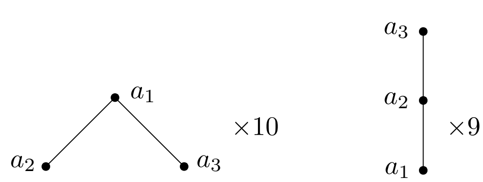


FIGURE 3. Plurality Counterexample

Acknowledgements. We benefitted greatly from conversations about this work with Bill Zwicker. We are also very grateful to the reviewers and editors, whose detailed comments and suggestions led to revisions and a new result (Theorem 2) that we feel significantly improved the paper.

REFERENCES

[1] M. Ackerman, S. Choi, P. Coughlin, E. Gottlieb, J. Wood, Elections with partially ordered preferences. To appear in *Public Choice*.

[2] K. Arrow, A Difficulty in the Concept of Social Welfare, The Journal of Political Economy 58 (4), 328-346 (1950)

[3] J.P. Barthélemy, Arrow's theorem:unusual domain and extended codomain, Mathematical Social Sciences, 3, 79-89 (1982)

[4] G. Brightwall, P. Winkler, Counting linear extensions, Order 8, 225-242 (1991)

[5] J.D. Brown, An approximate solution to Arrow's problem, Journal of Economic Theory, 9 (4), 375-383 (1974)

[6] J.D. Brown, Aggregation of preferences, Quarterly Journal Economics, 89 (3), 456-469 (1975)

[7] P.Y. Chebotarev, E. Shamis, Characterizations of scoring methods for preference aggregation, Annals of Operations Research, 80, 299-332 (1998)

[8] R. Fagin, R. Kumar, M. Mahdian, D. Sivakumar, E. Vee, Comparing and aggregating rankings with ties. In: Proceedings of the 2004 ACM Symposium on Principles of Database Systems, 47-58, (2004)

[9] J.A. Ferejohn, P.C. Fishburn, Representation of binary decision rules by generalized decisiveness structures, Journal of Economic Theory, 21, 28-45 (1979)

[10] R. Niemi, The Problem of Strategic Behavior under Approval Voting, American Political Science Review, 78, 952-958 (1984)

[11] F. Roberts, B. Tesman, Applied Combinatorics, 2nd Ed. Pearson, New Jersey (2005)

[12] D. Saari, Basic geometry of voting. Springer-Verlag, Berlin (1995)

[13] R. Stanley, Enumerative Combinatorics, Vol. 1, Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge (1997)

[14] D. Woodall, Properties of Preferential Election Rules, Voting Matters, 3, 8-15 (1994)

[15] H.P. Young, An Axiomatization of Borda's Rule, Journal of Economic Theory, 9, 43-52 (1974)