Symplectic stabilizers with applications to abelian varieties

John Cullinan

Department of Mathematics, Bard College
Annandale-On-Hudson, NY 12504
cullinan@bard.edu

Abstract

Fix a prime number $\ell > 2$ and let V be a 4-dimensional $\mathbf{F}_\ell$-vector space. We classify subgroups G of $\mathrm{Sp}(V)$ with the property that every $g \in G$ stabilizes a 1-dimensional subspace of V, yet G itself is not of parabolic type. This classification is motivated by its application to the mod ℓ representation of abelian surfaces, following a program outlined by Sutherland.

Keywords: local-global; isogeny; abelian variety

1 Introduction

Let K be a number field and fix a prime number ℓ. Sutherland has recently shown [10] that if E is an elliptic curve defined over K that admits a local-ℓ isogeny almost everywhere, then E admits a local-ℓ isogeny over an extension K' of K of degree ≤ 2. He arrives at this result by analyzing the subgroup structure of $\mathrm{GL}_2(\mathbf{F}_\ell)$ and comparing it with the possible images of the mod ℓ representation attached to E. In particular, he finds an example of an elliptic curve over $\mathbf{Q}$ that admits a local-7 isogeny almost everywhere but not globally and shows it is the only (up to isomorphism) counterexample over $\mathbf{Q}$. In this paper, we are interested in extending these results to higher-dimensional abelian varieties.

Let A be a d-dimensional ($d > 0$) abelian variety defined over K and let

$$\overline{\rho_\ell} : \mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{Aut}(A[\ell])$$

be the associated mod ℓ representation. Let S be a set of good primes $\mathfrak{p}$ for A of density 1 and let $A_\mathfrak{p}$ denote the reduction of A modulo $\mathfrak{p}$. Suppose $A_\mathfrak{p}$ admits an ℓ-isogeny over the residue field $\mathbf{F}_\mathfrak{p}$ for all $\mathfrak{p} \in S$. Group-theoretically, this means that every $g \in \mathrm{Aut}(A[\ell])$ stabilizes a line in $A[\ell]$, viewed as a $2d$-dimensional $\mathbf{F}_\ell$-vector space. Equivalently, the characteristic polynomial of g has an $\mathbf{F}_\ell$-rational root. On the other hand, in order that A admit an ℓ-isogeny over K, it is necessary that $\mathrm{Gal}(\overline{K}/K)$ stabilize a line in $A[\ell]$, *i.e.* that $\mathrm{im}\,\overline{\rho_\ell}$ be contained in an appropriate parabolic subgroup of $\mathrm{Aut}(A[\ell])$.

It is a more difficult problem to determine whether there exists an abelian variety defined over K with the specified mod ℓ representation. For example, let E be an elliptic curve over $\mathbf{Q}$ and take $\ell = 13$. The subgroup $2.A_4 \simeq \mathrm{SL}_2(\mathbf{F}_3)$ of $\mathrm{GL}_2(\mathbf{F}_{13})$ has the property that all of its elements have characteristic polynomials which split over $\mathbf{F}_{13}$, yet the mod 13 Galois representation is absolutely irreducible. However, there exists no elliptic curve defined over $\mathbf{Q}$ with the specified mod 13 Galois action since $2.A_4 \subset \mathrm{SL}_2(\mathbf{F}_{13})$ and the determinant map $\det : \mathrm{im}\,\overline{\rho_\ell} \longrightarrow \mathbf{F}_\ell^\times$ must be surjective for elliptic curves defined over $\mathbf{Q}$.

For the prime number $\ell = 2$, some results are known in higher dimensions. In order that every $g \in \mathrm{im}\,\overline{\rho_2}$ have an $\mathbf{F}_2$-rational eigenvalue it must be the case that $\det(g - 1) = 0$ in $\mathbf{F}_2$ for all $g \in \mathrm{im}\,\overline{\rho_2}$. The global condition is that the Jordan-Hölder series of $A[2]$ as a $\mathrm{Gal}(\overline{K}/K)$-module contain a trivial factor. Katz has shown [5] that this local-global principal holds for abelian surfaces, and in [3] it was shown to hold for threefolds. However, an example due to Serre [3] shows that the Steinberg representation of the simple group $\mathrm{L}_3(2)$ gives rise to an absolutely irreducible subgroup of $\mathrm{GSp}_8(\mathbf{F}_2)$ for which every element has 1 as an eigenvalue. We do not know whether there exists an abelian variety defined over $\mathbf{Q}$ with this mod 2 representation.

In this paper we take V to be a four-dimensional $\mathbf{F}_\ell$-vector space and classify the non-parabolic subgroups of $\mathrm{GSp}(V)$ for which every element has an $\mathbf{F}_\ell$-rational eigenvalue. Our main results are given in the following theorem (where "Type" refers to the classification scheme of [7]); the notations will be explained in the corresponding subsections of Section 3.

Theorem 1. *Let $\ell > 2$ be a prime number. The maximal irreducible subgroups G of $\mathrm{Sp}_4(\mathbf{F}_\ell)$ with the property that every $g \in G$ have an $\mathbf{F}_\ell$-rational eigenvalue are given in the following table:*

Type	*Group*	*Condition*
$\mathcal{C}_2$	$N_{\mathrm{GL}_2(\mathbf{F}_\ell)}(C_s)$	$\ell \equiv 1(4)$
	$(\ell-1)/2.\,\mathrm{SL}_2(\mathbf{F}_3).2$	$\ell \equiv 1(24)$
	$(\ell-1)/2.\,\mathrm{GL}_2(\mathbf{F}_3).2$	$\ell \equiv 1(24)$
	$(\ell-1)/2.\widehat{S_4}.2$	$\ell \equiv 1(24)$
	$(\ell-1)/2.\,\mathrm{SL}_2(\mathbf{F}_5).2$	$\ell \equiv 1(60)$
	$D_{(\ell-1)/2} \wr S_2$	
	$\mathrm{SL}_2(\mathbf{F}_3) \wr S_2$	$\ell \equiv 1(48)$
	$\widehat{S_4} \wr S_2$	$\ell \equiv 1(48)$
	$\mathrm{SL}_2(\mathbf{F}_5) \wr S_2$	$\ell \equiv 1(120)$
$\mathcal{C}_6$	$2_-^{1+4}.\mathrm{O}_4^-(2)$	$\ell \equiv 1(120)$
	$2_-^{1+4}.3$	$\ell \equiv 5(24)$
	$2_-^{1+4}.5$	$\ell \equiv 5(40)$
	$2_-^{1+4}.S_3$	$\ell \equiv 5(24)$
$\mathcal{S}$	$2.S_6$	$\ell \equiv 1(120)$
	$\mathrm{SL}_2(\mathbf{F}_5)$	$\ell \equiv 1(30)$
	$\mathrm{SL}_2(\mathbf{F}_3)$	$\ell \equiv 1(24)$

As stated above, one application of this group-theoretic classification is to abelian surfaces for which a certain local-global principle does not hold. It would be interesting to create "natural" examples of such surfaces (as opposed to starting with a surface with full Galois image and extending the field of definition). Examples of abelian fourfolds with O_8^+ Galois image have been constructed in [11] and [12].

2 Symplectic Groups and Abelian Varieties

As above, let A be a d-dimensional abelian variety defined over a number field K. Choosing a K-polarization on A, one gets a Galois-equivariant alternating form on the Tate module $T_\ell(A)$ with values in $\mathbf{Z}_\ell(1) := \varprojlim_n \mu_{\ell^n}(\overline{K})$; under reduction modulo ℓ, these values lie in $\mu_\ell(\overline{K})$. If the reduction of the alternating form is non-degenerate on $A[\ell]$, then the image of $\overline{\rho_\ell}$ is contained in the group of symplectic similitudes of $A[\ell]$. If the reduction is degenerate, then the kernel is an even-dimensional subspace W of $A[\ell]$ and there exists a non-degenerate alternating form on the quotient $A[\ell]/W$. For details, see [4, 9].

We take $d = 2$ so that for non-degenerate parings, a choice of basis for $A[\ell]$ yields $\mathrm{im}\,\overline{\rho_\ell} \subset \mathrm{GSp}_4(\mathbf{F}_\ell)$ and for degenerate parings $\mathrm{im}\,\overline{\rho_\ell} \subset \mathrm{GL}_2(\mathbf{F}_\ell) \times \mathrm{GL}_2(\mathbf{F}_\ell)$. We will not consider the case of degenerate pairings in this paper. Group theoretically, this case has essentially been analyzed in [10]. For the rest of this section, we take A to be principally-polarized.

The image of the mod ℓ representation

$$\overline{\rho_\ell} : \mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{Aut}(A[\ell])$$

then defines a subgroup $\mathscr{G}$ of $\mathrm{GSp}_4(\mathbf{F}_\ell)$. Suppose that for a set S of good primes $\mathfrak{p}$ of $\mathcal{O}_K$ for A of density 1, it is true that $A \mod \mathfrak{p}$ admits a local ℓ-isogeny over the residue field $\mathbf{F}_\mathfrak{p}$. Then $\mathscr{G}$ has the property that every element $g \in \mathscr{G}$ has an $\mathbf{F}_\ell$-rational eigenvalue [10]. If, in addition, $A[\ell]$ is an irreducible $\mathbf{F}_\ell[\mathscr{G}]$-module, then $\mathscr{G}$ is not contained in a parabolic subgroup of $\mathrm{GSp}_4(\mathbf{F}_\ell)$. We are interested in classifying subgroups of this type. The subgroup $\mathscr{G} \cap \mathrm{Sp}_4(\mathbf{F}_\ell)$ corresponds via Galois theory to enlarging the field K to contain the ℓth roots of unity. By enlarging K in this way, however, ℓ will be ramified.

If V is a 4-dimensional symplectic vector space, then any element of $\mathrm{Sp}(V)$ has eigenvalues of the form $a^{\pm 1}, b^{\pm 1}$. Moreover, any element of $\mathrm{GSp}(V)$ has eigenvalues of the form $\alpha, \beta, \lambda\alpha^{-1}, \lambda\beta^{-1}$. In terms of Galois reresentations, this is characterized by the fact that $\det \overline{\rho_\ell} = \overline{\chi_\ell}^2$, where $\overline{\chi_\ell}$ is the mod ℓ cyclotomic character. Thus, if $H \subset \mathrm{Sp}_4(\mathbf{F}_\ell)$ consists entirely of elements with an $\mathbf{F}_\ell$-rational eigenvalue, so does any overgroup of H contained in $\mathrm{GSp}_4(\mathbf{F}_\ell)$. We will henceforth turn our attention to subgroups of $\mathrm{Sp}_4(\mathbf{F}_\ell)$.

To begin enumerating the possibilities of $\mathscr{G}$, we will appeal to the subgroup structure of $\mathrm{Sp}_4(\mathbf{F}_\ell)$. Indeed, consider the intersection $\mathscr{G} \cap \mathrm{Sp}_4(\mathbf{F}_\ell)$. In terms of Galois theory, this amounts to replacing the underlying number field K with $K(\mu_\ell(\overline{K}))$. By Clifford's theorem, $A[\ell]$ is either an irreducible $\mathbf{F}_\ell[\mathscr{G} \cap \mathrm{Sp}_4(\mathbf{F}_\ell)]$-module, or is completely reducible and equipped with a transitive $\mathscr{G}/\mathscr{G} \cap \mathrm{Sp}_4(\mathbf{F}_\ell)$-action on the irreducible components. It is not true that every element of $\mathrm{Sp}_4(\mathbf{F}_\ell)$ has an $\mathbf{F}_\ell$-rational eigenvalue, so we will require the subgroup structure of $\mathrm{Sp}_4(\mathbf{F}_\ell)$. The non-parabolic maximal subgroups of $\mathrm{Sp}_4(\mathbf{F}_\ell)$ are given in the following table, using the notation of [6] and [7]:

Table 1: Non-parabolic maximal subgroups of $\mathrm{Sp}_4(\mathbf{F}_\ell)$

Type	**Group**	**Conditions**	**#Conjugates**
$\mathcal{C}_2$	$\mathrm{SL}_2(\mathbf{F}_\ell) \wr S_2$	$\ell \geq 3$	1
	$\mathrm{GL}_2(\mathbf{F}_\ell).2$	$\ell \geq 5$	1
$\mathcal{C}_3$	$\mathrm{SL}_2(\mathbf{F}_{\ell^2}).2$		1
	$\mathrm{GU}_2(\mathbf{F}_\ell).2$	$\ell \geq 5$	1
$\mathcal{C}_6$	$2_-^{1+4}.\Omega_4^-(2)$	$\ell \equiv \pm 3(8)$	1
	$2_-^{1+4}.\mathrm{O}_4^-(2)$	$\ell \equiv \pm 1(8)$	2
$\mathcal{S}$	$\mathrm{SL}_2(\mathbf{F}_\ell)$	$\ell \geq 7$	1
	$2.S_6$	$\ell \equiv \pm 1(12)$	2
	$2.A_6$	$\ell \equiv \pm 5(12)$	1

We are interested in subgroups of $\mathrm{Sp}_4(\mathbf{F}_\ell)$ with the following Property:

(L): Every $g \in G$ has an $\mathbf{F}_\ell$-rational eigenvalue.

In the next section we enumerate the maximal irreducible subgroups of $\mathrm{Sp}_4(\mathbf{F}_\ell)$ for which (L) holds.

3 Irreducible subgroups of $\mathrm{Sp}_4(\mathbf{F}_\ell)$

We split this section into subsections based on the type of the subgroup according to the classification scheme of [6, 7] and outlined in the table above.

3.1 Subgroups of Type $\mathcal{C}_2$

The subgroups of $\mathrm{Sp}_4(\mathbf{F}_\ell)$ occurring in this subsection are the stabilizers of certain subspace decompositions of the symplectic vector space $A[\ell]$. The decompositions can be non-degenerate or totally-singular. In dimension 4, each corresponds to a decomposition into two 2-dimensional subspaces. In the non-degenerate case, the stabilizer is isomorphic to $\mathrm{SL}_2(\mathbf{F}_\ell) \wr S_2$ and in the totally singular case it is $\mathrm{GL}_2(\mathbf{F}_\ell).2$.

The subgroup $\mathrm{GL}_2(\mathbf{F}_\ell).2$ fits into the split exact sequence

$$1 \longrightarrow \mathrm{GL}_2(\mathbf{F}_\ell) \longrightarrow \mathrm{GL}_2(\mathbf{F}_\ell).2 \xrightarrow{\pi} S_2 \longrightarrow 1$$

and, upon choosing a suitable basis, embeds into $\mathrm{Sp}_4(\mathbf{F}_\ell)$ in the following block form:

$$\left\{ \left(\begin{smallmatrix} A & 0 \\ 0 & A^{t-1} \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & B \\ -B^{t-1} & 0 \end{smallmatrix}\right) \;\middle|\; A, B \in \mathrm{GL}_2(\mathbf{F}_\ell) \right\}$$

We set the following notation: for any subgroup G of $\mathrm{GL}_2(\mathbf{F}_\ell).2$, define $G^0 := G \cap \ker \pi$. Thus G^0 defines a subgroup of $\mathrm{GL}_2(\mathbf{F}_\ell)$, whence G^0 either has order divisible by ℓ, is contained in the normalizer of a Cartan subgroup, or has projective image A_4, S_4, or A_5. We do not have to analyze the case where G^0 is contained in a non-split Cartan subgroup (since the eigenvalues are necessarily defined over a quadratic extension) or the case where G^0 is contained in a Borel subgroup, since then the action of G is reducible. The remainder of the cases are covered in the following Proposition.

Proposition 1. *The maximal irreducible subgroups G of $\mathrm{GL}_2(\mathbf{F}_\ell).2$ satisfying (L) are the following:*

$$\begin{aligned} N_{\mathrm{GL}_2(\mathbf{F}_\ell)}(C_s).2, &\qquad \ell \equiv 1(4) \\ (\ell-1)/2.\,\mathrm{SL}_2(\mathbf{F}_3).2, &\qquad \ell \equiv 1(24) \\ (\ell-1)/2.\,\mathrm{GL}_2(\mathbf{F}_3).2, &\qquad \ell \equiv 1(24) \\ (\ell-1)/2.\widehat{S_4}.2, &\qquad \ell \equiv 1(24) \\ (\ell-1)/2.\,\mathrm{SL}_2(\mathbf{F}_5).2, &\qquad \ell \equiv 1(60). \end{aligned}$$

Proof. If $A[\ell]$ is an irreducible $\mathbf{F}_\ell[G]$-module, then by Clifford's theorem $A[\ell]$ is a direct sum of two irreducible $\mathbf{F}_\ell[G^0]$-modules or remains irreducible upon restriction; due the embedding described above, the former occurs. Upon choosing a basis as above, this means that the projection of G^0 onto the plane spanned by the first two basis vectors defines an irreducible $\mathbf{F}_\ell[G^0]$-module. Furthermore, the condition that G satisfy (L) implies that G^0 does so also. Thus G^0 defines a subgroup of $\mathrm{GL}_2(\mathbf{F}_\ell)$ satisfying (L) whose natural action is irreducible.

As remarked above, it suffices to assume that G^0 does not normalize a non-split Cartan subgroup and that the order of G^0 is coprime to ℓ. So, suppose G^0 normalizes a split Cartan subgroup. Let $\mathbf{F}_\ell^\times = \langle \alpha \rangle$ and invoke the notation of [10, Prop. 3]: define

$$A(i,j) = \begin{pmatrix} \alpha^i & 0 \\ 0 & \alpha^j \end{pmatrix}, \qquad B(i,j) = \begin{pmatrix} 0 & \alpha^i \\ \alpha^j & 0 \end{pmatrix}.$$

Then G is partitioned into four subsets of the following type

$$G = \left\{ \begin{pmatrix} A(i,j) & 0 \\ 0 & A(i,j)^{t-1} \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 0 & A(i,j) \\ -A(i,j)^{t-1} & 0 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} B(i,j) & 0 \\ 0 & B(i,j)^{t-1} \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 0 & B(i,j) \\ -B(i,j)^{t-1} & 0 \end{pmatrix} \right\}$$

In order that the subset consisting of diagonal-block matrices of type $B(i,j)$ satisfy Property (L), it is necessary that i,j have the same parity. Moreover, every such $B(i,j)$ has $\mathbf{F}_\ell$-rational eigenvalues. For the off-diagonal subsets to have $\mathbf{F}_\ell$-rational eigenvalues, it is necessary that $\ell \equiv 1(4)$. Altogether, this yields a maximal irreducible subgroup of $\mathrm{Sp}_4(\mathbf{F}_\ell)$ of order $2(\ell-1)^2$ satisfying Property (L).

For any subgroup H of $\mathrm{GL}_2(\mathbf{F}_\ell)$, let PH be the image in $\mathrm{PGL}_2(\mathbf{F}_\ell)$ of the reduction of H modulo scalars. We now analyze the cases where PG^0 is isomorphic to A_4, S_4 or A_5. The Schur Multiplier of each group has order 2, so we may consider the lifts to $\mathrm{GL}_2(\mathbf{F}_\ell)$ in two stages: first a degree-2 lift, followed by a trivial central extension.

In each of these cases, the representation of G^0 that we are considering decomposes as a sum of a degree-2 representation and its dual. In this situation, the character of the induced representation of G is the *fusion join* [1, p. xxix] of the two degree-2 characters. The character values of the fusion join are simply the sums of the two character values that are being fused on the classes of G^0 and 0 on the classes of $G \setminus G^0$. By first lifting PG^0 by a degree-2 central extension, and then noting that any further central extension is trivial, we can deduce the characteristic polynomials and eigenvalues of the conjugacy classes. The maximal subgroups of $\mathrm{GL}_2(\mathbf{F}_\ell)$ that arise in this manner that also satisfy Property (L) are given by:

$$\begin{aligned} (\ell-1)/2.\,\mathrm{SL}_2(\mathbf{F}_3) &\qquad \ell \equiv 1(24) \\ (\ell-1)/2.\,\mathrm{GL}_2(\mathbf{F}_3) &\qquad \ell \equiv 1(24) \\ (\ell-1)/2.\widehat{S_4} &\qquad \ell \equiv 1(24) \\ (\ell-1)/2.\,\mathrm{SL}_2(\mathbf{F}_5) &\qquad \ell \equiv 1(60) \end{aligned}$$

A further lift to $\mathrm{GL}_2(\mathbf{F}_\ell).2$ preserves Property (L). If ℓ does not have the congruence properties listed above, then the maximal subgroups satisfying Property (L) are reducible. This completes the proof of the Proposition. $\square$

The other maximal subgroup of $\mathrm{Sp}_4(\mathbf{F}_\ell)$ of type $\mathcal{C}_3$ is isomorphic to $\mathrm{SL}_2(\mathbf{F}_\ell) \wr S_2$ and is the stabilizer of the decomposition of symplectic 4-space into two hyperbolic planes. With respect to a suitable basis, the embedding of $\mathrm{SL}_2(\mathbf{F}_\ell) \wr S_2$ is in block-matrix form, with the diagonal blocks representing the index-2 subgroup $\mathrm{SL}_2(\mathbf{F}_\ell) \times \mathrm{SL}_2(\mathbf{F}_\ell)$. We adopt the notation of above: for any subgroup G of $\mathrm{SL}_2(\mathbf{F}_\ell) \wr S_2$, let $G^0 = G \cap \ker \pi$, where π is the projection $\mathrm{SL}_2(\mathbf{F}_\ell) \wr S_2 \to S_2$. The following Proposition classifies the maximal irreducible subgroups of $\mathrm{SL}_2(\mathbf{F}_\ell) \wr S_2$ satisfying Property (L). We will invoke *Goursat's Lemma* [2, p. 864] on the subgroups of a direct product, which we now recall.

Lemma 1 (Goursat's Lemma)**.** *Let A and B be finite groups. The subgroups G of $A \times B$ are in one-to-one correspondence with the tuples (G_1, G_2, G_3, ψ) where $G_1 \leq A$, $G_2 \leq B$, $G_3 \lhd G_2$, and $\psi : G_1 \longrightarrow G_2/G_3$ is a surjective homomorphism.*

Proposition 2. *The maximal irreducible subgroups of $\mathrm{SL}_2(\mathbf{F}_\ell) \wr S_2 \subset \mathrm{Sp}_4(\mathbf{F}_\ell)$ satisfying Property (L) are the following:*

$$\begin{aligned} &D_{(\ell-1)/2} \wr S_2 \\ &\mathrm{SL}_2(\mathbf{F}_3) \wr S_2 \qquad (\ell \equiv 1(48)) \\ &\widehat{S_4} \wr S_2 \qquad (\ell \equiv 1(48)) \\ &\mathrm{SL}_2(\mathbf{F}_5) \wr S_2 \qquad (\ell \equiv 1(120)). \end{aligned}$$

Proof. The maximal subgroups of $\mathrm{SL}_2(\mathbf{F}_\ell) \wr S_2$ are, up to isomorphism:

$$B \wr S_2, \qquad D_{\ell-1} \wr S_2, \qquad D_{\ell+1} \wr S_2, \qquad E \wr S_2, \qquad \mathcal{G}, \qquad \mathrm{SL}_2(\mathbf{F}_\ell) \times \mathrm{SL}_2(\mathbf{F}_\ell),$$

where B is the maximal Borel subgroup of $\mathrm{SL}_2(\mathbf{F}_\ell)$; $D_{\ell\pm1}$ are the dihedral groups of orders $2(\ell \pm 1)$, respectively; E is any subgroup of $\mathrm{SL}_2(\mathbf{F}_\ell)$ with projective image A_4, S_4, or A_5; and $\mathcal{G}$ is the subgroup such that $\mathcal{G}^0$ corresponds to the quadruple

$$(\mathrm{SL}_2(\mathbf{F}_\ell), \mathrm{SL}_2(\mathbf{F}_\ell), \{\pm I\}, \pi),$$

where $\{\pm I\}$ is the center of $\mathrm{SL}_2(\mathbf{F}_\ell)$ and π is the projection $\mathrm{SL}_2(\mathbf{F}_\ell) \to \mathrm{PSL}_2(\mathbf{F}_\ell)$, afforded by Goursat's Lemma.

Of those maximal subgroups, we do not consider $B \wr S_2$ and $\mathrm{SL}_2(\mathbf{F}_\ell) \times \mathrm{SL}_2(\mathbf{F}_\ell)$ since the associated representation is reducible. Moreover, the subgroup $D_{\ell+1} \wr S_2$ is primarily comprised of elements without $\mathbf{F}_\ell$-rational eigenvalues (the non-trivial elements of the maximal cyclic subgroup of $D_{\ell+1}$ do not have $\mathbf{F}_\ell$-rational eigenvalues). The maximal subgroups of $D_{\ell+1} \wr S_2$ satisfying (L) are reducible. This leaves the groups $D_{\ell-1} \wr S_2$, $E \wr S_2$, and $\mathcal{G}$.

One can check, using an argument similar to the one in Proposition 1, that the maximal irreducible subgroup of $D_{\ell-1} \wr S_2$ satisfying (L) is isomorphic to $D_{(\ell-1)/2} \wr S_2$, where the index-2 dihedral groups are required to ensure that every element have an $\mathbf{F}_\ell$-rational eigenvalue.

Next, let $G \subset E \wr S_2$ and define $G^0 \subset E \times E$ via $G^0 := G \cap \ker \pi$, where π is the natural projection $E \wr S_2 \to S_2$. We have seen that the congruences $\ell \equiv 1(24)$, $\ell \equiv 1(24)$, and $\ell \equiv 1(60)$ are necessary and sufficient to ensure that the subgroups of $\mathrm{SL}_2(\mathbf{F}_\ell)$ in this category ($E \simeq \mathrm{SL}_2(\mathbf{F}_3)$, $\widehat{S_4}$, and $\mathrm{SL}_2(\mathbf{F}_5)$, respectively) satisfy (L). Any element $g \in G \setminus G^0$ has characteristic polynomial of the form

$$x^4 + bx^2 + 1,$$

with roots $\pm\lambda^{\pm1}$. Such a polynomial is reducible over $\mathbf{F}_\ell$, for all ℓ [8, Lemme 2.6]. To further ensure that all characteristic polynomials split, note that $g^2 \in G^0$ has eigenvalues $\lambda^{\pm2}$, each with multiplicity 2. Thus, for each group E, we must determine which congruences yield the largest subgroups satisfying (L). One can check that in the cases where $E \simeq \mathrm{SL}_2(\mathbf{F}_3), \widehat{S_4}$, the congruence $\ell \equiv 1(48)$ is necessary and sufficient for all elements of $E \wr S_2$ to have an $\mathbf{F}_\ell$-rational eigenvalue and that if $\ell \not\equiv 1(48)$ the maximal subgroups satisfying (L) are reducible. Similarly, in the case $E \simeq \mathrm{SL}_2(\mathbf{F}_5)$, taking $\ell \equiv 1(120)$ ensures that Property (L) is satisfied.

Finally, consider the subgroup $\mathcal{G}$ of $\mathrm{SL}_2(\mathbf{F}_\ell) \wr S_2$ described above. In this case $\mathcal{G}^0$ can be identified with the pairs $(g, \pm g)$, for $g \in \mathrm{SL}_2(\mathbf{F}_\ell)$, acting on the natural decomposition of symplectic 4-space into two hyperbolic planes. Clearly not every element has $\mathbf{F}_\ell$-rational eigenvalues. Moreover any maximal subgroup G of $\mathcal{G}$ is a subgroup of $D_{\ell-1} \wr S_2$, $D_{\ell+1} \wr S_2$, or $E \wr S_2$, with the property that $G^0 := G \cap \mathcal{G}^0$ is the restriction of π to $D_{\ell-1}$, $D_{\ell+1}$, or E. All of these subgroups have been subsumed by the cases above. This completes the proof of the Proposition. □

3.2 Subgroups of Type $\mathcal{C}_3$

The two maximal subgroups of $\mathrm{Sp}_4(\mathbf{F}_\ell)$ in this section are the so-called field extension subgroups of $\mathrm{Sp}_4(\mathbf{F}_\ell)$. Combining the field-extension embedding $\mathbf{F}_{\ell^2}^\times \hookrightarrow \mathrm{GL}_2(\mathbf{F}_\ell)$ with the symplectic transformations on a 4-dimensional vector space, one obtains maximal subgroups that fit into the middle of the short exact sequence

$$1 \longrightarrow G^0 \longrightarrow G^0.2 \longrightarrow \mathrm{Gal}(\mathbf{F}_{\ell^2}/\mathbf{F}_\ell) \longrightarrow 1,$$

where $G^0 \simeq \mathrm{SL}_2(\mathbf{F}_{\ell^2})$ or $\mathrm{GU}_2(\mathbf{F}_\ell)$; for details, see [7, Section 4.3]. The associated representation of $G^0.2$ is absolutely irreducible, but the restriction to the index-2 subgroup G^0 is not. The representation of G^0 decomposes over $\mathbf{F}_{\ell^2}$ into a direct sum of the natural representation of G^0 and its Galois-conjugate. The character values of the four-dimensional representation of G^0 are the sums of the values of the conjugate representations defined over $\mathbf{F}_{\ell^2}$. From these, we can determine the characteristic polynomials of the conjugacy classes. We will now show that the groups occurring in this subsection do not give rise to any new irreducible subgroups of $\mathrm{Sp}_4(\mathbf{F}_\ell)$ satisfying (L).

We start with the case $G^0 \simeq \mathrm{SL}_2(\mathbf{F}_{\ell^2})$. There are elements of G^0 that do not have $\mathbf{F}_\ell$-rational eigenvalues, hence any subgroup G of $\mathrm{SL}_2(\mathbf{F}_{\ell^2})$ satisfying (L) must have G^0 a subgroup of a Borel subgroup; $D_{\ell^2 \pm 1}$; a degree-2 central extension of A_4, S_4, or A_5; or isomorphic to $\mathrm{SL}_2(\mathbf{F}_\ell).2$.

The Jordan-Hölder factors of this four-dimensional representation of the Borel subgroup of $\mathrm{SL}_2(\mathbf{F}_{\ell^2})$ are all one-dimensional over $\mathbf{F}_\ell$, hence the four-dimensional representation of G is reducible over $\mathbf{F}_\ell$. The rotational subgroups of the dihedral groups only have $\mathbf{F}_\ell$-rational eigenvalues when those rotational subgroups are defined over $\mathbf{F}_\ell$. For D_{ℓ^2-1}, this means the maximal subgroups satisfying (L) have previously been analyzed in the proof of Propositions 1 and 2; for D_{ℓ^2+1}, the only elements with $\mathbf{F}_\ell$-rational eigenvalues are contained in the intersection with D_{ℓ^2-1}. When G^0 is an exceptional subgroup of $\mathrm{SL}_2(\mathbf{F}_{\ell^2})$ the four-dimensional representation splits into a direct sum of irreducible 2-dimensional representations over $\mathbf{F}_\ell$. These have been analyzed in Proposition 2.

This leaves the subgroup $\mathrm{SL}_2(\mathbf{F}_\ell).2$ of $\mathrm{SL}_2(\mathbf{F}_{\ell^2})$. We can write $\mathrm{SL}_2(\mathbf{F}_\ell).2 = \mathrm{SL}_2(\mathbf{F}_\ell) \cup \sigma\, \mathrm{SL}_2(\mathbf{F}_\ell)$ for some $\sigma \in \mathrm{SL}_2(\mathbf{F}_{\ell^2})$. Over $\mathbf{F}_{\ell^2}$ there exists a basis with respect to which $\mathrm{SL}_2(\mathbf{F}_\ell).2 \subset \mathrm{Sp}_4(\mathbf{F}_\ell)$ has the following form:

$$\left\{ \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}, \begin{pmatrix} \sigma g & 0 \\ 0 & \overline{\sigma} g \end{pmatrix} \ \mid g \in \mathrm{SL}_2(\mathbf{F}_\ell) \right\},$$

where $\overline{\sigma}$ denotes the Galois-conjugate of σ. This is because the modular character of natural representation of $\mathrm{SL}_2(\mathbf{F}_\ell)$ splits (in the sense of [1, p. xxviii]) for the group $\mathrm{SL}_2(\mathbf{F}_\ell).2$. Moreover, the eigenvalues of any representative σ as above are not in $\mathbf{F}_\ell$. Combining this with the facts that there are elements of $\mathrm{SL}_2(\mathbf{F}_\ell)$ without $\mathbf{F}_\ell$-rational eigenvalues and that eigenvalues of this 4-dimensional representation are simply the eigenvalues of the natural module with multiplicity 2, we conclude that this group will not yield any new examples of irreducible subgroups of $\mathrm{Sp}_4(\mathbf{F}_\ell)$ satisfying (L).

The other subgroup of $\mathrm{Sp}_4(\mathbf{F}_\ell)$ of type $\mathcal{C}_3$ that occurs is isomorphic to $\mathrm{GU}_2(\mathbf{F}_\ell).2$. Recall that $\mathrm{GU}_2(\mathbf{F}_\ell)$ is the isometry group of a 2-dimensional unitary space defined over $\mathbf{F}_{\ell^2}$. Moreover, $\mathrm{SU}_2(\mathbf{F}_\ell) \simeq \mathrm{SL}_2(\mathbf{F}_\ell)$

and $\det \mathrm{GU}_2(\mathbf{F}_\ell)$ is a cyclic subgroup of $\mathbf{F}_{\ell^2}^\times$ of order $\ell+1$. If δ is a generator of $\det \mathrm{GU}_2(\mathbf{F}_\ell)$, then $\mathrm{GU}_2(\mathbf{F}_\ell) \simeq \mathrm{SU}_2(\mathbf{F}_\ell){:}\langle\delta\rangle$ [7, p. 23].

As with $\mathrm{SL}_2(\mathbf{F}_{\ell^2})$, the 4-dimensional representation of $\mathrm{GU}_2(\mathbf{F}_\ell)$ that occurs here decomposes over $\mathbf{F}_{\ell^2}$ as direct sum of the natural representation of $\mathrm{GU}_2(\mathbf{F}_\ell)$ and its Galois-conjugate. Therefore, to determine the rationality of the eigenvalues of the 4-dimensional representation, it suffices to work with the standard representation only. Let $\widetilde{\delta} \in \mathrm{GL}_2(\mathbf{F}_\ell)$ denote a preimage of δ. Choose a basis and fix $\widetilde{\delta} = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then one can write

$$\mathrm{GU}_2(\mathbf{F}_\ell) = \bigcup_{n=0}^{\ell} \widetilde{\delta}^n\, \mathrm{SU}_2(\mathbf{F}_\ell).$$

It is not difficult to check that given $g \in \mathrm{SU}_2(\mathbf{F}_\ell)$, the eigenvalues of $\widetilde{\delta} g$ will not be $\mathbf{F}_\ell$-rational unless g is triangular. Moreover, $\mathrm{SU}_2(\mathbf{F}_\ell)$ contains elements whose eigenvalues are not $\mathbf{F}_\ell$-rational. Thus, if G^0 contains an element of order ℓ, it will be contained in the Borel subgroup of $\mathrm{SU}_2(\mathbf{F}_\ell)$ and the resulting representation of G will be reducible. Moreover, if $G^0 \cap \mathrm{SU}_2(\mathbf{F}_\ell)$ contains any non-triangular element, then G^0 (and hence G) will not satisfy (L). Altogether, this shows that any subgroup of type $\mathcal{C}_3$ does not give rie to any new irreducible subgroups satisfying (L).

3.3 Subgroups of Type $\mathcal{C}_6$

The subgroups of $\mathrm{Sp}_4(\mathbf{F}_\ell)$ of type $\mathcal{C}_6$ are the normalizers of symplectic-type 2-groups. In this case, the symplectic-type 2-group is of type

$$2_-^{1+4} \simeq D_4 \circ Q_8,$$

with normalizer isomorphic to $2_-^{1+4}.\Omega_4^-(2)$ when $\ell \equiv \pm 3$ (5) and $2_-^{1+4}.\mathrm{O}_4^-(2)$ when $\ell \equiv \pm 1(8)$. Recall the isomorphisms $\Omega_4^-(2) \simeq A_5$ and $\mathrm{O}_4^-(2) \simeq S_5$. We classify the irreducible subgroups having Property (L) in the following Proposition.

Proposition 3. *The maximal irreducible subgroups G of $\mathrm{Sp}_4(\mathbf{F}_\ell)$ of type $\mathcal{C}_6$ that satisfy Property (L) are given by:*

1. $G \simeq 2_-^{1+4}.\mathrm{O}_4^-(2)$ *when* $\ell \equiv 1(120)$
2. $G \simeq 2_-^{1+4}.3$ *when* $\ell \equiv 5(24)$
3. $G \simeq 2_-^{1+4}.5$ *when* $\ell \equiv 5(40)$
4. $G \simeq 2_-^{1+4}.S_3$ *when* $\ell \equiv 5(24)$

Proof. Suppose first that $\ell \equiv \pm 1(8)$ so that the maximal Type-$\mathcal{C}_6$ subgroup of $\mathrm{Sp}_4(\mathbf{F}_\ell)$ is $2_-^{1+4}.\mathrm{O}_4^-(2)$. The group $2_-^{1+4}.\mathrm{O}_4^-(2)$ has 25 conjugacy classes, consisting of elements of order 1, 2, 3, 4, 5, 6, 8, 10, 12, and 24. The congruence conditions on ℓ that ensure an $\mathbf{F}_\ell$-rational root of the characteristic polynomial of those classes are:

$$\ell \equiv 1(2),\ \ell \equiv 1(3),\ \ell \equiv 1(4),\ \ell \equiv 1(5),\ \ell \equiv \pm 1(24).$$

For example, the characteristic polynomials of the two classes of elements of order 24 are

$$x^4 - \sqrt{2}x^3 + x^2 - \sqrt{2}x + 1 \qquad \text{and} \qquad x^4 + \sqrt{2}x^3 + x^2 + \sqrt{2}x + 1,$$

where $\sqrt{2}$ denotes a fixed square-root of 2 modulo ℓ. Imposing the condition $\ell \equiv 1(120)$ means that all characteristic polynomials have an $\mathbf{F}_\ell$-rational root. Moreover, this 4-dimensional representation of the group $2_-^{1+4}.\mathrm{O}_4^-(2)$ is absolutely irreducible.

When $\ell \equiv \pm 3(8)$, we look to maximal subgroups of $2_-^{1+4}.\Omega_4^-(2)$. The conditions $\ell \equiv 1(4)$, $\ell \equiv 1(8)$, $\ell \equiv 1(12)$ which are necessary for the elements of order 4, 8, and 12 to have $\mathbf{F}_\ell$-rational eigenvalues are incompatible with $\ell \equiv \pm 3(8)$. In particular, when $\ell \equiv 3(8)$ a subgroup with Property (L) will not have elements of order 4, 8, or 12; when $\ell \equiv 5(8)$, they will not have elements of order 8.

There are four maximal subgroups of $2_-^{1+4}.\Omega_4^-(2)$:

$$\mathrm{SL}_2(\mathbf{F}_5), \qquad 2_-^{1+4}.S_3, \qquad 2_-^{1+4}.D_5, \qquad 2_-^{1+4}.A_4.$$

The 4-dimensional representation of the subgroup $\mathrm{SL}_2(\mathbf{F}_5)$ is not irreducible – it is a direct sum over $\mathbf{F}_\ell$ of two 2-dimensional representations, the other three maximal subgroups act irreducibly.

One can check that any subgroup of $2_-^{1+4}.S_3$ that omits elements of order 4, 8, and 12 when $\ell \equiv 3(8)$ or omits elements of order 8 when $\ell \equiv 5(8)$ is either abelian or dihedral and the associated representation is reducible.

The group $2_-^{1+4}.D_5$ has elements of order 8, so we look to its maximal subgroups. There are three maximal subgroups: $2.D_5$, $2^{1+4}.2$, and $2_-^{1+4}.5$. The first is reducible and the second contains elements of order 8. The third contains no elements of order 8, but does contain elements of order 4.

The group $2_-^{1+4}.A_4$ contains elements of order 4, 8, and 12. There are three maximal subgroups: $2_-^{1+4}.3$, $2_-^{1+4}.2^2$, and $2_-^{1+4}.S_3$. The first and third groups do not contain elements of order 8. The second group does contain elements of order 8 and its maximal subgroup without such elements is isomorphic to 2_-^{1+4}. In every case, the maximal subgroups without elements of order 4 are reducible. The congruence conditions on the groups then follow. □

3.4 Subgroups of Type $\mathcal{S}$

According to the table in Section 2 above, the subgroups $2.A_6$, $2.S_6$ and $\mathrm{SL}_2(\mathbf{F}_\ell)$ are those of type $\mathcal{S}$ occurring in the classification in [6, 7]. We begin with the groups $2.A_6$ and $2.S_6$. In ATLAS notation, there are two irreducible, symplectic characters of $2.A_6$, labeled χ_8 and χ_9. Each of these characters induces two distinct characters on $2.S_6$ whose values on $2.S_6 \setminus 2.A_6$ are negatives of each other (this is defined as a *splitting* of characters [1, p. xxviii]). However, only χ_8 splits into symplectic characters, while χ_9 splits into two characters of indicator 0. The symplectic characters of $2.S_6$ will be labeled χ_8^0 and χ_8^1 using ATLAS notation. The characteristic polynomials of the conjugacy classes are given in Tables 2 and 3 (for the conjugacy classes $6B_0$ and $6B_1$ of $2.S_6$, $\sqrt{3}$ denotes a fixed square root of 3, guaranteed to exist since $\ell \equiv \pm 1(12)$).

Lemma 2. *The group $2.A_6$ contains no irreducible subgroups with Property (L).*

Proof. In order that $2.A_6$ occur as a maximal subgroup of $\mathrm{Sp}_4(\mathbf{F}_\ell)$, it must be the case that $\ell \equiv \pm 5(12)$. On the other hand, in order that each characteristic polynomial above have an $\mathbf{F}_\ell$-rational root, it is necessary that $\ell \equiv 1(120)$, and these two sets of primes are disjoint. In light of this observation, it suffices to examine proper subgroups of $2.A_6$ with the property that they have no elements of order 3 when $\ell \equiv 5(12)$ and no elements of order 4 when $\ell \equiv 7(12)$.

If $\ell \equiv 5(12)$, then the maximal non-cyclic subgroups of $2.A_6$ of order dividing 80 are $\mathbf{Z}/5 \rtimes \mathbf{Z}/4$ and $Q_8.2$ (the Sylow-2 subgroup of $2.A_6$). The group $\mathbf{Z}/5 \rtimes \mathbf{Z}/4$ satisfies (L) for any such ℓ, but the Jordan Hölder series of the associated representation consists of four 1-dimensional factors. Imposing the additional condition $\ell \equiv 1(8)$ is necessary for $Q_8.2$ to satisfy (L), but in that case the representation splits into two absolutely-irreducible 2-dimensional factors.

Next suppose that $\ell \equiv 7(12)$ and recall that $2.A_6 \simeq \mathrm{SL}_2(\mathbf{F}_9)$. The maximal subgroups of $2.A_6$ are (up to isomorphism):

$$\mathbf{Z}/9 \rtimes \mathbf{Z}/8, \qquad \mathrm{SL}_2(\mathbf{F}_3).2, \qquad \mathrm{SL}_2(\mathbf{F}_5).$$

The maximal subgroup of $\mathbf{Z}/9 \rtimes \mathbf{Z}/8$ that does not contain elements of order 4 is simply $\mathbf{Z}/9 \rtimes \mathbf{Z}/2$. Any 4-dimensional representation of this group will split into a direct sum of two modules of dimension 2 (and possibly further) over $\mathbf{F}_\ell$ and so we do not consider it. We may similarly recursively dismiss the groups $\mathrm{SL}_2(\mathbf{F}_3).2$ and $\mathrm{SL}_2(\mathbf{F}_5)$. This completes the proof of the Proposition. □

Lemma 3. *When $\ell \equiv 1(120)$ the group $2.S_6$ satisfies Property (L). When $\ell \equiv -1(12)$, then no irreducible subgroup of $2.S_6$ satisfies (L).*

Proof. The group $2.S_6$ is a maximal subgroup of $\mathrm{Sp}_4(\mathbf{F}_\ell)$ when $\ell \equiv \pm 1(12)$. First, note that the condition $\ell \equiv 1(120)$ is necessary for all characteristic polynomials to have an $\mathbf{F}_\ell$-rational eigenvalue, and $2.S_6$ is absolutely irreducible.

When $\ell \equiv -1(12)$, then the cyclotomic polynomials $\varphi_3(x)$, $\varphi_4(x)$, $\varphi_6(x)$, do not have an $\mathbf{F}_\ell$-rational root. Moreover, the square of any element of order 8 has characteristic polynomial $\varphi_4(x)^2$. Therefore, any subgroup satisfying (L) must have order dividing 160 and contain no elements of order 4. The only such subgroup of $2.S_6$ is cyclic of order 10. Moreover, if we impose the condition $\ell \equiv 1(5)$ to ensure that every element have an $\mathbf{F}_\ell$-rational eigenvalue, then this group is reducible. This completes the proof of the Lemma. □

Table 2: Characteristic Polynomials of $2.A_6$

Conjugacy Class of $2.A_6$	χ_9	χ_8
$1A_0$	$\varphi_1(x)^4$	$\varphi_1(x)^4$
$1A_1$	$\varphi_2(x)^4$	$\varphi_2(x)^4$
$2A_0$	$\varphi_4(x)^2$	$\varphi_4(x)^2$
$3A_0$	$\varphi_3(x)^2$	$\varphi_3(x)^2$
$3A_1$	$\varphi_6(x)\varphi_2(x)^2$	$\varphi_6(x)^2$
$3B_0$	$\varphi_3(x)^2$	$\varphi_1(x)^2\varphi_3(x)$
$3B_1$	$\varphi_6(x)^2$	$\varphi_6(x)\varphi_2(x)^2$
$4A_0$	$\varphi_8(x)$	$\varphi_8(x)$
$4A_1$	$\varphi_8(x)$	$\varphi_8(x)$
$5A_0$	$\varphi_5(x)$	$\varphi_5(x)$
$5A_1$	$\varphi_{10}(x)$	$\varphi_{10}(x)$
$5B_0$	$\varphi_5(x)$	$\varphi_5(x)$
$5B_1$	$\varphi_{10}(x)$	$\varphi_{10}(x)$

Table 3: Characteristic Polynomials of $2.S_6$

Conjugacy Class of $2.S_6$	χ_8^0	χ_8^1
$1A_0$	$\varphi_1(x)^4$	$\varphi_1(x)^4$
$1A_1$	$\varphi_2(x)^4$	$\varphi_2(x)^4$
$2A_0$	$\varphi_4(x)^2$	$\varphi_4(x)^2$
$3A_0$	$\varphi_3(x)^2$	$\varphi_3(x)^2$
$3A_1$	$\varphi_6(x)\varphi_2(x)^2$	$\varphi_6(x)^2$
$3B_0$	$\varphi_3(x)^2$	$\varphi_1(x)^2\varphi_3(x)$
$3B_1$	$\varphi_6(x)^2$	$\varphi_6(x)\varphi_2(x)$
$4A_0$	$\varphi_8(x)$	$\varphi_8(x)$
$5A_0$	$\varphi_5(x)$	$\varphi_5(x)$
$5A_1$	$\varphi_{10}(x)$	$\varphi_{10}(x)$
$2B$	$\varphi_1(x)^2\varphi_2(x)^2$	$\varphi_1(x)^2\varphi_2(x)^2$
$2C$	$\varphi_4(x)^2$	$\varphi_4(x)^2$
$4B$	$\varphi_8(x)$	$\varphi_8(x)$
$6A_0$	$\varphi_3(x)\varphi_6(x)$	$\varphi_3(x)\varphi_6(x)$
$6A_1$	$\varphi_3(x)\varphi_6(x)$	$\varphi_3(x)\varphi_6(x)$
$6B_0$	$\varphi_4(x)(x^2-\sqrt{3}x+1)$	$\varphi_4(x)(x^2+\sqrt{3}x+1)$
$6B_1$	$\varphi_4(x)(x^2+\sqrt{3}x+1)$	$\varphi_4(x)(x^2-\sqrt{3}x+1)$

Proposition 4. *The maximal irreducible subgroups of* $\mathrm{SL}_2(\mathbf{F}_\ell)$ *satisfying (L) are* $\mathrm{SL}_2(\mathbf{F}_5)$ *when* $\ell \equiv 1(30)$ *and* $\mathrm{SL}_2(\mathbf{F}_3)$ *when* $\ell \equiv 1(24)$.

Proof. The embedding $\mathrm{SL}_2(\mathbf{F}_\ell) \hookrightarrow \mathrm{Sp}_4(\mathbf{F}_\ell)$ is via the symmetric-power representation Sym^3. Of the $\ell+4$ conjugacy classes of $\mathrm{SL}_2(\mathbf{F}_\ell)$, $(\ell-3)/2$ of them (consisting of elements of order $\ell+1$) have eigenvalues defined over a quadratic extension of $\mathbf{F}_\ell$. If an element $g \in \mathrm{SL}_2(\mathbf{F}_\ell)$ has eigenvalues $\lambda^{\pm 1}$, then $\mathrm{Sym}^3(g)$ has eigenvalues $\lambda^{\pm 3, \pm 1}$. It follows that if the eigenvalues of g are not defined over $\mathbf{F}_\ell$, then neither are those of

$\mathrm{Sym}^3(g)$. Therefore, the exceptional group $\mathrm{SL}_2(\mathbf{F}_\ell)$ does not satisfy the hypothesis that all of its elements stabilize a line in $\mathbf{F}_\ell^4$, for any ℓ.

If we turn to the subgroup structure of $\mathrm{SL}_2(\mathbf{F}_\ell)$, we can omit parabolic subgroups from our analysis since the symmetric power representation will take a parabolic subgroup of $\mathrm{SL}_2(\mathbf{F}_\ell)$ to one of $\mathrm{Sp}_4(\mathbf{F}_\ell)$. Moreover, if G is dihedral then $\mathrm{Sym}^3(G)$ decomposes into a direct sum of two-dimensional irreducible representations for G and these have been analyzed previously in [10]. This leaves the exceptional groups $2.A_4$, $2.S_4$ and $2.A_5$.

Again, we exploit the isomorphisms $2.A_4 \simeq \mathrm{SL}_2(\mathbf{F}_3)$ and $2.A_5 \simeq \mathrm{SL}_2(\mathbf{F}_5)$. The representation Sym^3 is reducible for $\mathrm{SL}_2(\mathbf{F}_3)$ and absolutely irreducible for $\mathrm{SL}_2(\mathbf{F}_5)$. Thus, it suffices to consider only $\mathrm{SL}_2(\mathbf{F}_5)$, which occurs as a maximal subgroup of $\mathrm{SL}_2(\mathbf{F}_\ell)$ when $\ell \equiv \pm 1(10)$.

If $\ell \equiv 1(10)$, then imposing the additional condition that $\ell \equiv 1(3)$ is necessary and sufficient for all characteristic polynomials to split over $\mathbf{F}_\ell$. The absolute irreducibility of $\mathrm{Sym}^3 \mathrm{SL}_2(\mathbf{F}_5)$ gives us an obstruction. On the other hand, if $\ell \equiv -1(10)$, then no element of $\mathrm{SL}_2(\mathbf{F}_5)$ of order 5 has $\mathbf{F}_\ell$-rational eigenvalues. The maximal subgroups of $\mathrm{SL}_2(\mathbf{F}_5)$ of index divisible by 5 are $\mathbf{Z}/5 \rtimes \mathbf{Z}/4$, D_6, and $\mathrm{SL}_2(\mathbf{F}_3)$, and Sym^3 of each of these groups is reducible.

This leaves the subgroup $2.S_4 \simeq \mathrm{SL}_2(\mathbf{F}_3).2 \subset \mathrm{SL}_2(\mathbf{F}_\ell)$. This group has one absolutely irreducible 4-dimensional representation and it is precisely the symmetric-power representation Sym^3. The group $2.S_4$ is a maximal subgroup of $\mathrm{SL}_2(\mathbf{F}_\ell)$ when $\ell \equiv \pm 1(8)$. If $\ell \equiv 1(8)$ and we impose the additional condition that $\ell \equiv 1(3)$, then every characteristic polynomial splits over $\mathbf{F}_\ell$. On the other hand, if $\ell \equiv -1(8)$, then the characteristic polynomials of the elements of order 8 do not split over $\mathbf{F}_\ell$ and any subgroup with no elements of order 8 is reducible under Sym^3. This proves the Proposition. □

Acknowledgements. We would like to thank Yuri Zarhin for helpful comments.

References

[1] J. Conway *et. al.*, *ATLAS of Finite Groups*, Oxford University Press, Cambridge, 1985.

[2] G. Butler and J. McKay, The transitive groups of degree up to eleven. *Comm. Algebra.* **11** (1983), 863-911.

[3] J. Cullinan, A computational approach to the 2-torsion structure of abelian threefolds. *Math. of Comp.* **78** (2009), 1825-1836.

[4] E. Howe, Isogeny classes of abelian varieties with no principal polarizations. In: *Moduli of abelian varieties (Texel Island, 1999)*, 203–216, Progr. Math., 195, Birkhäuser, Basel, 2001.

[5] N.M. Katz, Galois properties of torsion points on abelian varieties. *Invent. math.* **62** (1981), 481-502.

[6] P. Kleidman, *The Subgroup Structure of some Finite Simple Groups*, Ph.D. thesis, Trinity College, Cambridge, 1987.

[7] P. Kleidman and M. Liebeck, *The Subgroup Structure of the Finite Classical Groups*, Cambridge University Press, 1990.

[8] P. LeDuff, Representations galoisiennes associés aux points d'ordre ℓ des jacobiennes de certaines courbes de genre 2. *Bulletin de la S.M.F.* tome **126** no. 4 (1998), 507-524.

[9] A. Silverberg and Yu. G. Zarhin, Polarizations on abelian varieties. *Math. Proc. Cambridge Philos. Soc.* **133** (2002), no. 2, 223–233.

[10] A. Sutherland, A local-global principle for rational isogenies of prime degree. Preprint available at `http://arxiv.org/abs/1006.1782`.

[11] A. Vàrilly-Alvarado, D. Zywina, Arithmetic E_8 lattices with maximal Galois action. *LMS J. Comput. Math.* **12** (2009), 144-165.

[12] Yu. G. Zarhin, Del Pezzo surfaces of degree 1 and Jacobians. *Math. Ann.* **340** (2008), no. 2, 407-435.