# PRIMES OF PRESCRIBED CONGRUENCE CLASS IN SHORT INTERVALS

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ABSTRACT. Suppose $k \leq 72$ is a positive integer and $a$ is an integer coprime to $k$. We show that for $x \geq 106706$, the interval $(x, 1.048x]$ contains a prime congruent to $a$ modulo $k$.

We are often in need of primes that fit specific requirements. Sometimes we want them to be of a certain size. The prototype of a result ensuring the existence of primes of the right size is the Bertrand Postulate, a classical result first proved in 1850 by Chebyshev; it states that for $x \geq 1$, the interval $(x, 2x]$ contains a prime. For a wonderful proof, see [1]. Refinements of this result show that for each $\delta > 0$, there is a constant $B(\delta)$ such that for $x \geq B(\delta)$, $(x, x + \delta x]$ contains a prime.

For some purposes, it is not the *size* of the prime that matters so much as its *remainder* upon division by some fixed integer. In this optic, the celebrated theorem of Dirichlet ensures that for co-prime integers $a, k$, the congruence class $a \bmod k$ contains infinitely many primes.

For some applications in number theory, one needs an amalgam of both types of results: we often need primes of the right size *and* of the right congruence class. Estimating the size of the smallest prime in the congruence class $a \bmod k$ for arbitrary $a$ and $k$ is a very subtle problem, intimately related to the Riemann Hypothesis. In this short note, we focus on a less demanding problem, namely,

*What can one say about the existence of primes of prescribed congruence classes in short intervals?*

We use standard techniques from analytic number theory (See *e.g.* [2], [5]) and an easily manageable computation to show how to obtain effective existence results of this type for small moduli $k$. For example, we prove that for $x \geq 7$, the interval $(x, 2x]$ contains a prime congruent to $1 \bmod 4$ as well as a prime congruent to $3 \bmod 4$.

**Theorem 1.** *Suppose $1 \leq k \leq 72$, and $a$ is any integer coprime to $k$. If $x \geq 106706$, or more precisely if $x \geq N(k)$ where $N(k)$ is given in Table 1, then the interval $(x, 1.048x]$ contains a prime congruent to $a \bmod k$.*

*Proof.* Fix a positive integer $k \leq 72$. We define

$$\theta(x; k, a) = \sum_{\substack{p \equiv a(k) \\ p \leq x}} \log p,$$

where the sum is over the primes $p$ not exceeding $x$ in the congruence class $a \bmod k$. The interval $(x, y]$ contains a prime in the congruence class $a \bmod k$ if and only if $\theta(y; k, a) - \theta(x; k, a) > 0$.

Our proof relies on the explicit estimates of Ramaré-Rumely [5], in which two uniform bounds are given for $\theta(x; k, a)$ in the ranges $x \geq 10^{10}$ and $x < 10^{10}$. We begin by assuming $x \geq 10^{10}$. In [5, Thm. 1], the authors obtain the bound

$$\max_{1 \leq y \leq x} \left| \theta(y; k, a) - \frac{y}{\varphi(k)} \right| \leq \epsilon \frac{x}{\varphi(k)} \tag{1}$$

for $x \geq 10^{10}$ where $\epsilon = 0.023269$ and $\varphi$ is Euler's phi function (see [5, Table 1]). Applying (1) twice, once with parameter $(1 + \delta)x$ and then again with parameter $x$, we find

$$\theta(x(1 + \delta); k, a) - \theta(x; k, a) \geq ((1 - \epsilon)(1 + \delta) - (1 + \epsilon)) \frac{x}{\varphi(k)}.$$

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To show the left hand side is positive, it suffices to ensure that $\delta - 2\epsilon - \epsilon\delta > 0$, i.e.

$$\delta > \frac{2\epsilon}{1-\epsilon}.$$

The latter holds as long as $\delta > 0.04765$. In particular, we have shown that for $x \geq 10^{10}$, $(x, 1.048x]$ contains a prime congruent to $a$ mod $k$.

For $x \leq 10^{10}$, we can appeal to [5, Thm. 2]:

$$\max_{1\leq y\leq x} \left| \theta(y;k,a) - \frac{y}{\varphi(k)} \right| \leq 2.072\sqrt{x}, \qquad 1 \leq x \leq 10^{10}. \tag{2}$$

Applying (2) with parameter $1.048x$ as well as with $x$, we find

$$\theta(1.048x;k,a) - \theta(x;k,a) \geq \frac{.048x}{\varphi(k)} - 2.072(\sqrt{1.048x} + \sqrt{x}).$$

The right hand side is positive as long as

$$\sqrt{x} > \frac{2.072(1+\sqrt{1.048})\varphi(k)}{.048}.$$

Thus, we have shown that

$$\left(\frac{259(5+\sqrt{131/5})\varphi(k)}{30}\right)^2 < x \leq 10^{10} \implies$$
$$\theta(1.048x;k,a) - \theta(x;k,a) > 0.$$

Since $k \leq 72$, we have $\varphi(k) \leq 70$, hence $\theta(1.048x;k,a) - \theta(x;k,a) > 0$ as soon as $x \geq 37393267$. The rest of the proof is a finite computation which we carried out in PARI/GP [4]. Namely, we verify that for each $k \leq 72$, and each integer $x$ in the range

$$N(k) \leq x \leq \left(\frac{259(5+\sqrt{131/5})\varphi(k)}{30}\right)^2,$$

the interval $(x, 1.048x]$ contains primes of every eligible congruence class modulo $k$. Here, $N(k)$ is the optimal lower bound for each modulus $k$, as listed in Table 1. The largest $N(k)$ for $k \leq 72$ occurs for $k = 71$ and has value $N(71) = 106706$, completing the proof of the theorem. Our program in GP/PARI used the tables of primes incorporated into the package and proceeded by reducing all primes in the stated interval modulo $k$ to ensure that all eligible residue classes modulo $k$ were represented. □

*Remarks.*

(1) We note that Ramaré and Rumely [5, Thm. 1] provide effective estimates for quite a few other moduli $k$, including, for example all composite integers $k \in [73, 112]$. They also give better values of $\epsilon$ for larger lower bounds on $x$, which allow one to obtain similar statements about existence of primes of given residue class in $(x, x+\delta x]$ for smaller values of $\delta$. For instance, if $k \leq 72$, one obtains the result that $(x, 1.0175x]$ contains a prime in any congruence class $a$ mod $k$ with $a$ co-prime to $k$, as long as $x \geq 10^{100}$.

(2) In [3], Kadiri shows how to obtain a bound $N(k)$ as above for any $k$ which is "non-exceptional," meaning for which one can prove an appropriate zero-free region for all Dirichlet $L$-functions of conductor $k$. Kadiri gives tables of $N(k)$ only for large $k$, viz. $k \geq 5 \cdot 10^4$. By contrast, our emphasis here is on small moduli $k$.

(3) Table 1 below can be considered a refinement of the table computed by Harborth and Kemnitz in [2].

| $k$ | $N(k)$ | $k$ | $N(k)$ | $k$ | $N(k)$ | $k$ | $N(k)$ |
|---|---|---|---|---|---|---|---|
| 1 | – | 19 | 18246 | 37 | 32049 | 55 | 49236 |
| 2 | 213 | 20 | 5061 | 38 | 18246 | 56 | 24437 |
| 3 | 532 | 21 | 8559 | 39 | 22398 | 57 | 47421 |
| 4 | 887 | 22 | 6156 | 40 | 11272 | 58 | 24229 |
| 5 | 1793 | 23 | 23503 | 41 | 44330 | 59 | 70736 |
| 6 | 532 | 24 | 4859 | 42 | 8559 | 60 | 19902 |
| 7 | 3732 | 25 | 18538 | 43 | 45475 | 61 | 75246 |
| 8 | 2169 | 26 | 7856 | 44 | 20498 | 62 | 43683 |
| 9 | 3103 | 27 | 13962 | 45 | 23542 | 63 | 53072 |
| 10 | 1793 | 28 | 10364 | 46 | 23503 | 64 | 44320 |
| 11 | 6156 | 29 | 24229 | 47 | 73003 | 65 | 56097 |
| 12 | 1792 | 30 | 6429 | 48 | 13883 | 66 | 18534 |
| 13 | 7856 | 31 | 30271 | 49 | 60715 | 67 | 80335 |
| 14 | 3732 | 32 | 16501 | 50 | 18538 | 68 | 30194 |
| 15 | 6429 | 33 | 18534 | 51 | 30648 | 69 | 46621 |
| 16 | 5589 | 34 | 11593 | 52 | 27454 | 70 | 32040 |
| 17 | 11593 | 35 | 32040 | 53 | 68864 | 71 | 106706 |
| 18 | 3103 | 36 | 7013 | 54 | 13963 | 72 | 26463 |

Table 1

## References


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[4] PARI/GP, version `2.5.0`, Bordeaux, 2011, `http://pari.math.u-bordeaux.fr/`.

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