Voting with Partially-Ordered Preferences

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Abstract

This module introduces a method – the partial Borda count – for voting with partially-ordered preferences. The partial Borda count allows voters to vote with their true preferences rather than having a linear preference order imposed on the ballot structure *a priori*. Furthermore, this method provides a simple and mathematically consistent way of scoring truncated ballots and improves upon so-called *bullet voting* and other methods of resolving incomplete ballots. The mathematical content of the module is elementary combinatorics with applications to social choice theory.

We present the material for the classroom following an inquiry-based format with detailed examples and notes for the instructor. The take-home message for the student is threefold: there are inherent social problems with requiring linearly-ranked ballots; there is a simple procedure for allowing an arbitrary preference ranking to be expressed; and that method is the most desirable among all possible procedures for voting with partially-ordered preferences.

Keywords: social choice theory, voting preferences, Borda Count, partially-ordered sets, combinatorics, inquiry based learning, problem based learning, constructivist instruction.

The mathematical theory of voting with three or more alternatives has a rich history, going back at least to the mid 18^{th} century. Most of the well-known results in this subject are based on the assumption that the voters have linearly ranked the alternatives. But what if a voter's true preferences involve ties or more complicated partial orders?

The goal of this module is to explore a method of voting with non-linear preferences, as well as some of the method's implications for social justice. The mathematical techniques are self-contained and, depending on the level of the course, completely elementary. In particular, we introduce partial orders and use Hasse diagrams to represent ballots. The combinatorial properties of the posets then have implications for fairness properties of the partial Borda count voting system. From point of view of social justice, the main contribution is to show that it is possible to incorporate arbitrary preference rankings into a ballot that satisfies all standard fairness axioms; that is, if one's true preferences are non-linear, then that voter would not be forced to submit a linearly ranked (and therefore disingenuous) ballot. The partial Borda count voting system also produces a simple, mathematically consistent way to score a truncated ballot. All of these topics will be covered in detail in the following sections.

1 Mathematical content

Before beginning an increasingly detailed description of the mathematics, we will address three preliminary questions:

1. **What courses would this module work for?**

   This module is designed for an introductory class on voting theory or, more broadly, a general education or mathematics for liberal arts course as an activity on the mathematics of voting. It fits naturally into a semester-long course or into a survey course in which voting theory makes up one section or learning module.

2. **What are the prerequisite mathematical ideas?**

   The prerequisites for the students are modest. They should have familiarity with basic high school algebra and the instructor should review the following concepts:

   - Examples of Social Choice Functions (voting procedures), especially the Borda Count;
   - Fairness Criteria in Social Choice Theory.

   The purpose of this module is twofold: to introduce the notion of voting with partially-ordered preferences via the partial Borda count, and to show that the partial Borda count is, in a concrete sense, the mathematically "fairest" procedure for aggregating the ballots.

3. **What will be the mathematical value of the module?**

   This module will strengthen the students' techniques of mathematical reasoning and introduce basic combinatorial principles, such as Hasse diagrams and real-valued functions on posets. While these topics seem esoteric for an introductory class, one of the purposes of this module is to demonstrate that through the lens of voting theory, they are actually quite concrete and have real-world significance.

The specific mathematical content that we believe to be new for the student is reviewed in Appendix A.1 (the new-for-instructor content is summarized in Section 3). In this section we are more concerned with the pure mathematics than with the applications to social choice theory. Therefore, we postpone definitions of the relevant fairness axioms until Section 4, and a review worksheet for social choice theory is provided in Appendix A.2.

This material is meant to be developed in the context of a class that focuses on voting theory and social choice theory and assumes no mathematical prerequisites. In fact, the course we use this module in has only modest prerequisites (a passing score on our institution's math diagnostic test, i.e., proficiency in high school algebra). Individual instructors can choose, depending on the level of the students, how much mathematical formalism to include. In our course, after introducing standard voting techniques for three or more alternatives over several classes, we spend a day on voting with partially-ordered preferences as a way to generalize linear preferences. The techniques and equations are intuitive and work well in both standard and inquiry-based formats.

By the time students attempt the module, they should be familiar with the basics of social choice theory. In particular, students should have worked on examples or homework involving the Borda count and some other voting methods for comparison (Instant Runoff Voting, Plurality, among others). They should also have seen some of the fairness criteria used to assess social welfare functions, such as Pareto, the Condorcet Winner Criterion, and Independence of Irrelevant Alternatives [8]. Ideally, this module would come at the end of the social choice theory section of the course. Instructors should be familiar with social choice theory and basic set theory/combinatorics. In particular, the instructor will need to explain the basic properties of (Hasse diagrams of) posets and adapt the fairness criteria usually reserved for linearly ordered ballots to partially-ordered ones.

2 Context / Background

2.1 Institutional Context

This module has been used by the first author at Bard College during a course on mathematics and politics – a general-education course with no mathematical prerequisites. Bard College is a small liberal-arts college with approximately 2000 undergraduate students. The primary mission of the college is undergraduate education and, with the exception of a handful of Masters programs, is solely concerned with the teaching of undergraduate students. Historically, Bard students have tended to major more in the arts and literature than in science and mathematics. However, one of the tenets of a liberal-arts education is breadth of knowledge, and there has always been a general-education mathematics requirement.

While some students use Precalculus, Calculus, and beyond to satisfy the mathematics requirement, there are students at the college who lack the background for these courses. Rather than teach general survey courses, each mathematics instructor at Bard has designed their own semester-long general-education course based on a single theme for non-majors (some examples include *Explorations in Number Theory*, *Chance*, *Secret Codes*, and *Mathematics and Politics*.) This is the context in which this module was created.

Introductory courses on voting theory exist in many colleges and universities throughout the United States and, in addition, many general mathematical survey courses have voting theory components to them as well. The course at Bard College typically runs with 25-30 students.

The project to design a voting procedure that allowed for partially-ordered preferences began as a year-long senior thesis at Bard College. We then collaborated to generalize those results, in particular the uniqueness theorems, and the final product is [5]. The first author of this module uses this material in the classroom, much in the way it is presented below.

In 2013, we collaborated with a Masters student in Curatorial Studies at Bard College to use the partial Borda count to determine which art would be displayed in a curatorial exhibit. Over 200 people across campus (students, faculty, and staff) were interviewed about their art preferences and asked to rank, using partial orders if applicable, their true feelings on six aspects of an art installation. We then aggregated the ballots using the partial Borda count to help determine which aspects of the collection would be most visible to the public.

2.2 Underlying Social Justice Context

Almost all of the standard voting procedures (e.g., the Borda Count, Instant Runoff Voting, etc.) assume that a voter is able to rank alternatives linearly. But what if one's true feelings are more complicated – tree-like or non-linear? What if one is apathetic? Then the most-used voting procedures force voters to submit disingenuous ballots. Thus, we are interested in whether it is possible to create a new procedure, subject to standard fairness criteria in social choice theory, that allow for partially-ordered preferences. In particular, the fairness criteria should specialize to standard ones if all voters submit linearly ordered preferences.

A voting tactic known as *bullet voting* [7] is when a voter ranks only a subset of the alternatives and submits an incomplete ballot. (In fact, bullet voting specifically refers to the practice of voting for one alternative instead of a ranked ballot.) This practice is widely seen as a way to manipulate election outcomes because bulleted ballots carry more weight for the single alternative than in a fully complete ranked ballot. The partial Borda count provides a way to renormalize a bulleted ballot and score it with the others in a mathematically consistent way.

This module contributes to a conversation on social justice because it introduces a voting paradigm where voter preferences can be represented more accurately than in many of the other well-known methods. Our underlying assumption is that voting methods that represents voter preferences as truthfully as possible will make democracies more representative, more egalitarian, and fundamentally more just.

3 Instructor Preparation

Voting with partially-ordered preferences has been mentioned in the literature going back at least to Arrow's seminal work [2] with generalizations in [3] and [6]. The introduction of [5] gives some historical background and references and [1] serves as a gentle introduction to voting with partially-ordered preferences. In addition, Young's paper [10] (on which [5] was based) introduces a few different fairness criteria that are not typically discussed in introductory textbooks. Finally, we mention [4] for an encyclopedic reference on the many examples of social choice functions for partially-ordered profiles and the fairness axioms they satisfy.

We have written the teaching module with the idea that the class has covered the basic definitions of voting theory, whether as part of a semester-long course or as part of a longer module in the middle of a survey course. We have included a handout in Appendix A that we use in our course and give it to the students just prior to the beginning of the section of partial orders. In that handout we use colloquial language to complement the class text [8] and we find it is well-received by students as a study guide. For this paper, we assume the instructor is familiar with the usual fairness criteria used to evaluate social welfare functions (e.g., Pareto, etc.)

The module will introduce posets and their Hasse diagrams. While we do not use much of the theory of partially-ordered sets (we use posets and their Hasse diagrams interchangeably), we recommend that the instructor review their basic properties. We recommend [8] for the background on social choice theory and [9] for the combinatorics and posets.

4 The Module

We assume that this module comes during a section on social choice theory, whether as part of an entire semester course in mathematics and politics, or simply a section in a survey course. As presented, the teaching module is likely too long for a one-hour class but could easily fit into two such classes. Alternatively, the material could be amended to fit into a single class of an hour and twenty minutes (the length of a typical class at Bard College). Our approach is primarily inquiry-based and problem-based, though is easily adaptable to more traditional teaching formats. As stated in Section 1, we assume that the audience (students) is familiar with the basic definitions and language of social choice theory. See Appendix A.2 for a brief overview of prerequisite material in social choice theory; the overview could easily be used as a handout in class for the students.

We have written this module with the intent that it take place over two 80-minute class periods, comprising a typical week of classroom time at the authors' institution. Before beginning in detail, we give a very quick overview of the module and what we will focus on in this section.

1. **Review of the Borda count and Approval Voting.** We suggest beginning with a quick review of the Borda count and introducing the notion of poset by recalling the definition of approval voting and showing how approval ballots can be visualized via a Hasse diagram.

2. **Inquiry-based approach to posets.** We will discuss specific examples below, but we prefer to use leading questions in class to have the students come up with the definition of poset and Hasse diagram. We find that because of the mathematical background of our typical audience, the students absorb the material better than if we start by defining posets on the board. Once the students are comfortable with the idea, we suggest asking them to generate interesting, fun examples from everyday life in order to further solidify the basic properties of partial orders.

3. **The partial Borda count.** We will then show how we introduce the main topic of the module once the students have a good grasp of partial orders. Again, our approach is to let the students explore the material on their own with minimal direction from the instructor, though with well-chosen illustrative examples as guides. Once several key examples have been discussed, we share our recommendations for explaining the scoring formulas for the partial Borda count.

4. **Fairness.** Because fairness is a key issue for linearly-ranked ballots, it should be highlighted in the context of partially-ordered ballots as well. But how should we amend the standard fairness criteria to work for voting with partially-ordered preferences? That is the key question that we address in the final portion of the module.

4.1 Review of the Borda Count

We suggest beginning with a quick review of the Borda count. In order to make the partial Borda count which will be introduced agree with the classic Borda Count when a linearly ordered ballot is submitted, we recommend scaling the Borda scoring by a factor of 2 (this will not change the outcome of an election or the fairness properties of the voting procedure). That is, given a ballot of m alternatives, we assign a score of $2m-2$ to the most-preferred alternative, $2m-4$ to the second most-preferred, etc., ending with a score of 0 for the least-preferred alternative. For example, given the ballot

$$\begin{array}{c} a \\ | \\ b \\ | \\ c \\ | \\ d \\ | \\ e \end{array}$$

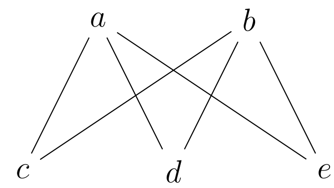
the alternatives a, b, c, d, and e would receive 8, 6, 4, 2, and 0 points respectively. Each ballot is scored according to this rule, and the alternative with the highest total score is declared the winner. It is important to point out to students that ties are possible, in which case the subset tied for the highest total score are all declared the social choices.

From the point of view of fairness, the Borda Count satisfies the 'Always a Winner' criterion (since it always produces a non-empty subset of the winners as the social choice(s)) as well as Pareto and Monotonicity [8, p. 13]. However, the Borda Count does not satisfy the Condorcet Winner Criterion (it may be the case that a Condorcet winner exists and the Borda Count would select a different alternative as the winner) or Independence of Irrelevant Alternatives. To keep this exposition self-contained, we will not discuss these two particular fairness metrics further, but rather refer to [8] for more information.

4.2 Approval Voting

Next we suggest a quick review of approval voting. Below is a narrative replicating what we might say in our classes.

> *A common method of voting when there are three or more alternatives is* ***approval voting*** *– for example, the American Mathematical Society uses approval voting for elections for the Nominating Committee and the Editorial Boards Committee [11, p. 1073]. Approval Voting is the procedure whereby a voter partitions the alternatives into two subsets: those which are approved, and those which are not; no further refinements of the approved subset are given. Then, the alternative who is approved most often (or a subset tied for most approvals) is declared the social choice. This naturally leads to the idea of a partial order (details and definitions are given in Appendix A.1 above) and is a gentle introduction to voting with partially-ordered preferences. For example, given alternatives* a, b, c, d*, and* e*, suppose a voter approves of* a *and* b *only. Then those preferences can be represented graphically as*



Indeed, any such ballot would be similarly partitioned, the approved subset lying above the unapproved, and edges indicating preference.

The main purpose of this module is to explore voting with arbitrary partial orders. In the next subsection we describe how instructors can introduce these ideas more formally in their classes. For more details and further development of approval voting, see [7].

4.3 Posets and preferences

We find that when first introducing the notion of a poset, it helps to have students both generate their own examples and to come up with the notion of the Hasse diagram as a visual description of the partial on their own, with no preliminary definitions from the instructor. Below we list a few sample introductory questions that can be used:

1. How would you graphically represent the following preferences among three ice cream flavors: *Chocolate is preferred to both vanilla and strawberry, but there is no preference between vanilla and strawberry.*

2. Everyone in the class individually think of your own preferences. How would you represent them graphically?

For the first question we expect to see an answer such as Figure 1(a). For the second question, we might elicit a few responses from the students in the class. Here are some possible responses:

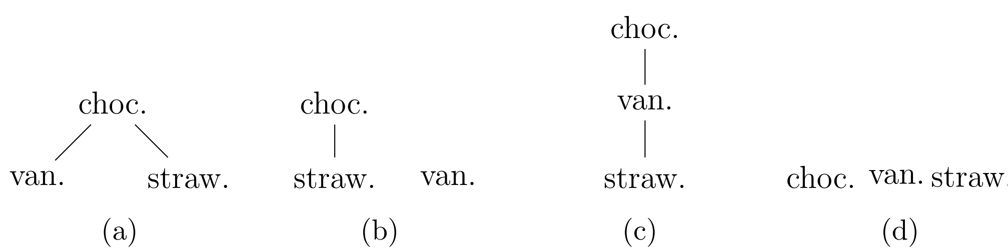


Figure 1: Some possible preferences among three flavors of ice cream.

While this may seem simple, we have found in practice that it takes several examples for an entire classroom to be comfortable with the notion of partially-ordered preferences, even for posets of size 3.

3. What are some favorable properties of partial orders?

This is a hard question for the students since they have almost no exposure to the subject beforehand, but we are trying to motivate the students to come up with the following properties that characterize posets:

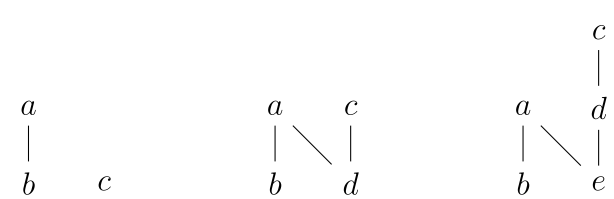
- An alternative cannot be preferred over itself;
- if a is preferred over b and b over c, then a is preferred over c.

It may also be instructive to point out the limitations of using posets to express preferences. For example, someone who loves all three flavors and has no preferences among them would use the same diagram, Figure 1(d), as someone who is merely apathetic. The *strength* of one's preference for any particular alternative is not directly encoded in a poset (nor is it in the traditional Borda count).

4.4 The partial Borda count

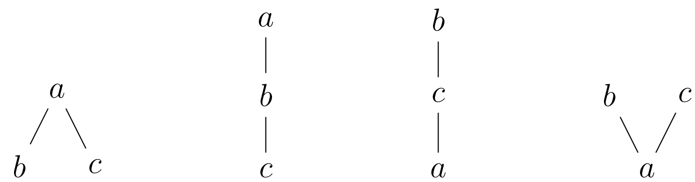
Who should win an election? This is the guiding question for this portion of the module and our goal is to have the students engage with it *before* giving them the formula for scoring the partially-ordered ballots. To start, we present a sample profile of ballots – perhaps continuing with the ice cream example – without giving information on how to score them, and ask students who they think should win. This makes students have to consider the role of antichains and singletons before giving away the formula from the partial Borda count and will make the formula more meaningful when it is presented. Specifically, we break this part of the class into the following steps.

Step 1. First, we give students a single poset and ask how they would determine the winner(s) if that were the only ballot. We do this with a few different types of posets of increasing complexity. For example, the following posets could make for interesting discussion:



The goal here is to get students thinking about how much weight to give alternatives based on their relative position to other alternatives. In the poset shown in the middle, for example, it is reasonable to say that a should win because it is preferred over more alternatives than c, even though a and c are at the same "level" in the poset.

Step 2. Next, we give an example of a profile of ballots and ask the students who the overall winner should be. We try to keep it simple enough so that it is fairly obvious who comes out on top on each ballot, but put a variety of ballots so it is not so clear who the overall winner should be, as in this example:



In this profile, it is clear that the first two ballots show preference for a over the others, the third ballot shows preference for b over the others, and the fourth ballot shows equal preference for b and c over a. If these were the only four ballots submitted, who should win? It may not be possible to come with a satisfactory resolution in class until the partial Borda count is presented. In this step our goal is just to get students thinking about what features of the posets should matter when aggregating ballots to determine the social choice. (It turns out that b is the winner if we score using the partial Borda count.)

Step 3. Third, we present the class with a single poset with three alternatives and tell them they have 6 points to distribute among the alternatives, with the more preferred alternatives getting more points. How should this be done? We then repeat with different types of posets.

For example, in a three-element antichain it should be clear that every alternative should get 2 points.

What about a chain? With a chain, say $a > b > c$, one would want to assign the most points to a, fewer points to b, and even fewer to c. There are several ways to do this, but if one wants to enforce the (reasonable) property that the point differential between a and b should be the same as that between b and c, there are just two possibilities: $(3, 2, 1)$ or $(4, 2, 0)$.

To decide, it helps to look at yet another example.

Consider the poset with relations $a > b$, $a > c$. (This is the first of the four posets shown in Step 2.) For this poset it is reasonable to assign equal points to b and c by symmetry; moreover the number of points assigned to a in this poset should arguably equal the number of points assigned to a in the chain $a > b > c$, because in both examples a is directly preferred over all the other alternatives. If a were given 3 points, there would be no way to equally divide up the remaining 3 points among b and c in the poset $a > b$, $a > c$.

By this reasoning the most plausible assignment of points to the chain $a > b > c$ would be $(4, 2, 0)$.

Step 4. Finally, we reveal Equation (1) and discuss its interpretation in terms of "giving points away" to more preferred alternatives. We then use this equation to calculate weights and the social choice for the profile used in Step 2 and present additional examples as needed.

4.5 Fairness and social justice implications

We believe that some discussion of the (mathematical) fairness properties are essential to the social justice component of the module, and so below we provide a very quick background that could easily be turned into a handout for the class.

4.5.1 Background on fairness criteria

Given a voting procedure, there are several standard metrics by which to evaluate it, often called **fairness properties**. **Arrow's theorem** states that the only voting procedure that satisfies Pareto, Independence of Irrelevant Alternatives, and Monotonicity is a dictatorship (i.e., choose a ballot at random and select the highest-ranked alternative). Similarly, the **Gibbard-Satterthwaite Theorem** shows that a dictatorship is the only voting procedure for three or more alternatives that does not produce ties for the social choice, satisfies Pareto, and is non-manipulable. For more information see [8, Ch. 7].

Even though a dictatorship satisfies the standard fairness criteria, it is not a socially acceptable voting procedure. So, we necessarily must give up some notions of fairness in order to hold an election. In [10], Young proved that the Borda Count is the unique voting procedure for linear ballots that is simultaneously **neutral**, **consistent**, **faithful**, and has the **cancellation property**. For precise definitions of these terms see [10] or [5], though for ease of exposition we give quick explanations:

- **Neutral.** Each alternative has an equal chance of winning before voting begins (i.e., if the order of the alternatives is permuted before voting, then the social choice(s) are permuted accordingly).
- **Consistent.** If disjoint sets of voters would choose the same social choice(s), then the union of the sets would as well.
- **Faithful.** If there is only one voter, and that voter prefers b over a, then a is not a winner.
- **Cancellation Property.** If, for all pairs of alternatives (a, b), the number of ballots where a is preferred to b equals the number where b is preferred to a, then a tie among all alternatives is declared.

It can further be shown that these properties imply the Borda Count is **monotone** and satisfies Pareto, two well-known and important properties for voting procedures. Likewise, in [5] the partial Borda count is shown to be the unique social choice function that is consistent, faithful, neutral, and has the cancellation property. This generalizes Young's result to include profiles of partially-ordered ballots, with no restrictions on the partial order.

4.5.2 Suggestions for the Classroom

Our major goal for the classroom is now to address the following question: Why is the partial Borda count a reasonable voting system? Based on the prerequisites laid out in Section 1 and Appendix A.1, we feel comfortable posing the following questions to the class. For each fairness criterion, we have the students break into groups and try to answer the following questions about that criterion.

- How would you phrase this criterion so it makes sense for partially-ordered preferences?
- Does the partial Borda procedure satisfy this criteria?

In most standard introductions to mathematics and politics and voting theory, the concepts of monotonicity and Pareto are thoroughly developed and we tend to focus on these two concepts and using the following approach to explaining them in class. Again, we break the students into groups and have them work together to answer the following questions:

1. How would you generalize the Pareto condition to partially-ordered ballots? What aspects of Pareto are the most salient, and what should be carried over to the context of partial orders? For partial orders, the Pareto condition carries over naturally:

   Pareto condition: *For two alternatives a, b, if every voter prefers b over a, then a is not a winner (i.e., not in the social choice set).*

2. What does monotonicity mean in the context of partial orders? This is more delicate, so we suggest beginning with examples where there are two alternatives with arbitrary partial orders, so the only possibilities are chains and antichains. For partial orders, the monotonicity condition must be altered slightly to accommodate incomparable elements.

   Monotonicity condition: *Let p be a profile and let a be a social choice. Suppose one of the voters changes her original preference order from $\preccurlyeq$ to a preference order $\preccurlyeq'$ with the property that for all $b, c \in A - \{a\}$,*

   $$b \prec c \Longleftrightarrow b \prec' c, \quad b \prec a \Longrightarrow b \prec' a, \quad \text{and} \quad a \not\prec b \implies a \not\prec' b$$

   Then a remains a social choice for the new profile p'.

By focusing on these two criteria, already known to the students from linear ballots, the nuances of partial orders can be explored while always having the colloquial context of voting to explain the results.

Finally, we suggest showing that the partial Borda count does indeed satisfy Pareto and Monotonicity. At that point we simply give a brief overview of the further properties enjoyed by the partial Borda count (listed above). The uniqueness result is beyond the scope of this module, but we feel it is worth stating to students for completeness.

4.5.3 Social justice implications

Now that the students have worked through the mathematics and have a good internalized idea of partially-ordered preferences, we bring up the social justice implications for the partial Borda count. First, the partial Borda count allows for the most general types of preferences to be allowed on a ballot and has been mathematically proven to be the unique such voting method that satisfies the standard fairness axioms. That is, by allowing arbitrary partial orders to be submitted on ballots, we do not force the voters to submit disingenuous preference orders if their true feelings are non-linear.

Second, the partial Borda count allows us to improve upon the drawbacks of the tactic of *bullet voting* [7]. Bullet voting is what occurs when a voter is supposed to submit a linearly-ordered ballot, but only submits one alternative or a proper subset of alternatives. Do we score that ballot or discard it? If we discard a large number of bullet-ballots, then we are not accurately portraying the collective choice of the voters. More generally, what if a voter submits a partially complete linearly-ordered ballot? How do we evaluate these ballots along with those that are fully complete?

The partial Borda count provides a simple answer that scores the bulleted ballots along with the completed ballots in a mathematically consistent way: place all the unscored alternatives into a single antichain and then proceed via the partial Borda count. This way, it is not possible to use the bullet voting tactic to game the original voting system.

5 Additional Thoughts

5.1 Application notes

The first author has taught this module five times in five iterations of a semester-long introduction to mathematics and politics. The material is extremely well-received by the students, and they tend to find it a more fulfilling conclusion to social choice theory than Arrow's theorem (and indeed, we have presented it in parallel with Arrow's theorem). We recommend taking the appropriate time to allow the students to generate their own examples of posets. An initial time investment in this aspect of the class helps solidify their intuition once the equations are introduced.

5.2 Extending the module

There is a significant increase in the level and technicality of the mathematics involved in so-called "fairness" properties of voting procedures, both for traditional linearly-ordered ballots and for partially-ordered ones as well. Because of the leap in sophistication and taking the intended audience's preparation into account, we omitted many of the details from that part of the module. The intent of this module is to comprise one or two classes of a semester and working through the details of the proofs is likely outside the scope of such a presentation. However it is conceivable that interested instructors could expand upon this section and explore fairness criteria in more depth.

5.3 Possible variants

We have found this material to be amenable to student research projects at the sophomore and junior level. As an example, consider the following questions: Given an integral vector $(n_1, \ldots, n_k)$ such that $\sum_j n_j$ is even, 1) do the n_j represent the weights of a partially-ordered ballot and 2) if so, is there a unique such ballot? The answer to both questions is no, and we have worked with students to characterize the ballots that describe these surjectivity and injectivity questions.

Finally, even though the material has been presented for non-majors, we believe the module could be easily adapted for an upper-level course on combinatorics, possibly for a capstone course. The article [5] is self-contained and could be read by advanced students.

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Appendix

A Assignments and Handouts

A.1 Overview of the mathematical formalism

For completeness, we provide here a quick overview of the mathematical formalism of posets and its application to the partial Borda count. Parts of what follows can be used as a review handout for students.

The partial Borda count generalizes to partially-ordered sets the ordinary Borda Count for linear orders. Therefore, to give a meaningful treatment of the voting theory, the following mathematical topics should be introduced to the class in conjunction with the material on social choice theory.

A **poset** (short for **partially-ordered set**) is a set P together with a relation $<$ satisfying the following axioms:

- Irreflexivity: For all $x \in P$, $x \not< x$.
- Transitivity: For all $x, y, z \in P$, if $x < y$ and $y < z$ then $x < z$.

We write $x \leq y$ if $x < y$ or $x = y$. In the literature it is more common to see posets axiomatized in terms of the relation $\leq$, but for this module it is more natural to use $<$.

A relation $x < y$ in P is called a **cover relation**, and y is said to **cover** x, if there is no $z \in P$ such that $x < z < y$. Two elements of a poset are said to be **comparable** if one is less than the other. Otherwise they are **incomparable**. A poset in which any two elements are comparable is called a **chain**. A poset in which no two elements are comparable is called an **antichain**.

The standard way to visualize a poset is through its **Hasse diagram**, in which elements of the poset are drawn as nodes, and a line is drawn from one node x up to another node y whenever y covers x. Many examples of Hasse diagrams appear throughout Section 4 in the development of the module.

In the context of voting, a poset can be used to express a voter's prefences among a set A of alternatives. Thus for two alternatives $x, y \in A$, the statement "y is preferred over x" would be encoded by the relation $x < y$. The poset of preferences for an individual voter constitutes that voter's **ballot**, and the collection of ballots for all voters in an election is referred to as a **profile**.

The partial Borda count, as defined in [5] is based on assigning weights to alternatives in each ballot, and then aggregating the weights across all ballots and declaring the alternatives with the maximal weight to be the social choice. Weights are assigned to alternatives on individual ballots as follows. For a voter v whose ballot consists of some partial ordering $<_v$ of the alternatives A, the weight given to an alternative $a \in A$ on this particular ballot is

$$w(a) = 2 \cdot d(a) + i(a) \tag{1}$$

where $d(a)$ is the number of alternatives $b \in A$ such that $b <_v a$, and $i(a)$ is the number of alternatives that are incomparable with a. See Figure 2 for several ballots with weights calculated for each alternative. This method of assigning weights to alternatives and determining the social choice by aggregating weights is called the **partial Borda count.**

To understand Equation (1), suppose there are n alternatives and each is given $n - 1$ points. Now require every alternative to give one of its point away to each alternative that is preferred over it. The number of points that an alternative $a \in A$ ends up with is precisely $2 \cdot d(a) + i(a)$. As an example to illustrate all of the ideas surrounding posets and the partial Borda count, we present the following profile with scores attached.

A.2 Overview of social choice theory

Prior to the start of the class where students will work on the module, we provide them with a quick recap of the main ideas from Social Choice Theory. That recap is presented below as a handout for students.

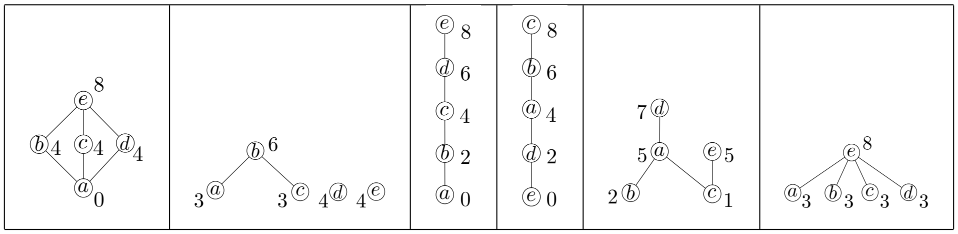


Figure 2: An Example of a Profile with Partial Borda Weights

Mathematics and Politics
Overview of Social Choice Theory

Voting with Two Alternatives. The main result in this area that we covered is **May's Theorem** and its generalization. In particular, if you want a social welfare function that is simultaneously

- Anonymous (it is impossible to tell who voted for whom);
- Neutral (each alternative has an equal chance of winning before voting begins); and
- Monotone (if alternative **A** wins, and someone changes her vote from **B** to **A**, then **A** still wins),

then you must use a **quota system**. That is, an alternative wins precisely when they get a certain percentage (quota) of the votes – between 50 and 100 percent – otherwise the election is declared a tie. When the quota just exceeds 50, we call this **majority rules**.

Voting with Three or More Alternatives. The situation is much more complicated when there are three or more alternatives and **Arrow's Theorem** shows the extent to which our inherent notions of fairness are challenged. The main examples of social choice procedures that we learned are the following.

The Condorcet Procedure

An alternative wins if and only if they beat everyone else in a one-on-one election; otherwise there is no winner. This is an extremely strong requirement and is hard to meet.

Plurality

The most first-place votes wins. This ignores all rankings of the alternatives in the ballots below the top choice.

Anti-Plurality

The fewest last-place votes wins. This ignores all rankings of the alternatives in the ballots above the last choice.

The Borda Count

Given n alternatives, assign the first-place of each ballot $n-1$ points, the second place $n-2$ points, *etc.* until the last place on each ballot gets 0 points. The alternative with the most points wins.

The Hare System

Remove the alternative(s) with the fewest first-place votes and rerun the election; repeat the procedure until there is only one alternative left or more than one tied. These are the social choices.

The Coombs System

Similar to anti-plurality, this voting procedure deletes the alternative(s) with the most last-place votes and then reruns the election. Repeat this procedure until there is one alternative left or more than one tied. These are the social choices.

Sequential Pairwise Voting with a Fixed Agenda

Choose an ordering of the alternatives. Then run an election (majority rules) between the first two. The winner moves on against the third, and so on. The winner is the alternative who is left at the end.

Dictatorship

Choose a ballot at random. The top choice on that ballot is the winner.

Fairness. In class we studied five basic fairness criteria: **Always a Winner**, **Pareto**, **Monotonicity**, **Independence of Irrelevant Alternatives**, and the **Condorcet Winner Criterion**. The voting procedures above all satisfy some of these criteria (the specifics are worked out in your text, or in the homework). **Arrow's Theorem** says that the only voting procedure that satisfies Pareto, IIA, and Monotonicity is a dictatorship.

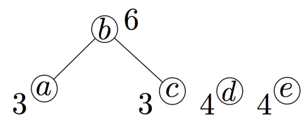
Other Examples.

Approval Voting

Each voter declares which alternatives they approve of. The most approvals wins.

The partial Borda count

Each voter submits a poset diagram of their true preferences (with ties and indecisiveness if applicable). The diagrams are then scored according to the following rule. If there are n alternatives, then each alternative is initially awarded $n-1$ points. Then each alternative must give one point to every other alternative that is ranked higher. An example with five alternatives is the following:



The alternative with the most points wins.

Lewis Carroll's example

The alternative with the fewest swaps of adjacent alternatives needed to become a Condorcet winner is declared the winner. The problem with this method is its practicality: the computational complexity of deciding a winner gets at some of the hardest problems in theoretical computer science.

Ostrogowski's Paradox

This example illustrates the paradoxical nature of voting for individual issues versus the party line that can easily occur in local elections, even when there are only two parties.

There are two parties: **D** and **R**; five voters: **1**, **2**, **3**, **4**, and **5**; and three issues: **Environment**, **Marriage Equality**, **Local Taxation**. Suppose each voter's true preferences are given in the following table.

Voter	Favorite on **Environment**	Favorite on **Marriage Equality**	Favorite on **Local Taxation**
1	**D**	**R**	**R**
2	**R**	**D**	**R**
3	**R**	**R**	**D**
4	**D**	**D**	**D**
5	**D**	**D**	**D**

If each voter votes for the party that best represents their views, then **R** wins, 3 to 2. On the other hand, **D** is the most-preferred party on all three issues.