LOCAL-GLOBAL PROPERTIES OF TORSION POINTS ON THREE-DIMENSIONAL ABELIAN VARIETIES

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Abstract. Let K be a number field and let $\ell > 5$ be a prime. We classify abelian threefolds A defined over K which have a non-trivial ℓ-torsion point mod $\mathfrak{p}$ for almost all primes $\mathfrak{p}$ of K, but are not K-isogenous to any abelian threefold over K with a non-trivial K-rational torsion point.

1. Introduction

Let A be an abelian variety defined over a number field K, and let S be a set of good primes $\mathfrak{p}$ for A of density 1; we write $A_{\mathfrak{p}}$ for the reduction of A modulo $\mathfrak{p}$ and $\mathbf{F}_{\mathfrak{p}}$ for the residue field at $\mathfrak{p}$. In [8], Katz considered the following question, originally posed by Lang:

Question 1 (**Lang**). *Let $m \geq 2$ be an integer. If $\#A_{\mathfrak{p}}(\mathbf{F}_{\mathfrak{p}}) \equiv 0(m)$ for all $\mathfrak{p} \in S$, does there exist a K-isogenous A' such that $\#A'(K)_{tor} \equiv 0(m)$?*

Lang's question is a converse of the property $A(K)[m] \hookrightarrow A_{\mathfrak{p}}(\mathbf{F}_{\mathfrak{p}})$ for any m prime to $\mathfrak{p}$, where $A(K)[m]$ is the K-rational kernel of the multiplication-by-m isogeny. In [8], Katz showed that Lang's question has a positive answer when A is an elliptic curve, and in the special case $m = \ell$ is prime, for two-dimensional abelian varieties. However, he exhibited explicit counterexamples in all dimensions greater than two.

The aim of this paper is to classify all counterexamples when $\dim A = 3$. To do this, we use the reformulation of this local-global problem given in [8] in terms of the group-theoretic properties of the image of the mod ℓ representation $\overline{\rho_\ell} : \mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{Aut}(T_\ell(A) \otimes \mathbf{F}_\ell)$. The fact that Question 1 is really a problem in group theory is one of the key consequences of Katz's paper.

Our counterexamples are either 1) comprised of groups of Lie type of smaller dimension, or 2) have order independent of ℓ. The latter category splits further: some are induced from Katz's original counterexamples in [8], while others are not. We call these counterexamples *Katz type* obstructions, and the rest *Exceptional type* obstructions. Our classification is subject to several hypotheses on $\overline{\rho_\ell}$ which will be explained in Section 4. The main result is as follows.

Theorem 1. *Let A be a three-dimensional abelian variety defined over a number field K and let S be a set of good primes for A of density 1. Suppose that $\ell \geq 7$, $\mathrm{im}\,\overline{\rho_\ell} \subset \mathrm{Sp}_6(\mathbf{F}_\ell)$, and that the projective image $\pi \circ \overline{\rho_\ell} : G_K \longrightarrow \mathrm{PSp}_6(\mathbf{F}_\ell)$ is not properly contained in the Hall-Janko group J_2. If $A_{\mathfrak{p}}(\mathbf{F}_{\mathfrak{p}})[\ell] \neq 0$ for all $\mathfrak{p} \in S$, and there exists no K-isogenous A' with $A'(K)[\ell] \neq 0$, then $\mathrm{im}\,\overline{\rho_\ell}$ is given in the following tables.*

Katz-Type	Lie-Type	Exceptional-Type
1. $\mathbf{Z}/2 \times \mathbf{Z}/2$	1. D_n, $D_n.2$	1. $2.S_4$, $2.S_4.2$ ($\ell \equiv \pm 1(8)$)
2. S_3, $S_3.2$	2. $\mathrm{SO}_3(\mathbf{F}_\ell), \mathrm{SO}_3(\mathbf{F}_\ell).2$	2. A_5, $A_5.2$ ($\ell \equiv \pm 1(10)$)
3. D_4, $D_4.2$	3. $\mathrm{SO}_3(\mathbf{F}_{\ell^2})$	
4. A_4, $A_4.2$	4. $\mathrm{PSL}_2(\mathbf{F}_\ell)$, $\mathrm{PSL}_2(\mathbf{F}_\ell).2$	
5. S_4, $S_4.2$	5. $\mathrm{GL}_2(\mathbf{F}_\ell).2$	
6. $2.S_3$	6. $\mathrm{SL}_2(\mathbf{F}_\ell) \rtimes \mathbf{Z}/3$	

Of all the counterexamples above, the most interesting are those of Lie type. Indeed, the Katz-type counterexamples are quite similar to the original, and the associated abelian varieties can be constructed almost identically. The Lie-type counterexamples, however, have the property that their orders depend on ℓ, and so are much larger subgroups of $\mathrm{Sp}_6(\mathbf{F}_\ell)$. Any geometric realization of a Lie-Type counterexample would certainly require different techniques.

Notation and Terminology. In this paper, K denotes a number field and $\overline{K}$ a fixed algebraic closure; denote by G_K the absolute Galois group $\mathrm{Gal}(\overline{K}/K)$. For an abelian variety A/K, let $T_\ell(A)$ be the ℓ-adic

Tate module of A, with associated ℓ-adic and mod ℓ representations ρ_ℓ and $\overline{\rho_\ell}$, respectively; the Weil pairing on $T_\ell(A)$ ensures that $\operatorname{im}\overline{\rho_\ell} \subset \mathrm{GSp}_6(\mathbf{F}_\ell)$. We call $G \subset \mathrm{GSp}_6(\mathbf{F}_\ell)$ an *obstruction* if $G = \operatorname{im}\overline{\rho_\ell}$ for some abelian threefold which violates Question 1 with $m = \ell$.

We follow the standard group-theoretic convention and write $A.B$ for the middle term of a short exact sequence of groups with kernel A and quotient B. This notation will be recalled in section 3 of the paper.

2. Katz's Reformulation and Counterexamples

In the case where $m = \ell$ is a prime, Katz reformulates [8, p. 481-483] Lang's original question in representation-theoretic terms, as follows:

Question 2. *Let A be an abelian variety over a number field K. If for every $g \in G_K$ we have $\det(1-\overline{\rho_\ell}(g)) = 0$ in $\mathbf{F}_\ell$, is it true that the semisimplification of $T_\ell(A)\otimes\mathbf{F}_\ell$ contains the trivial representation?*

We now summarize Katz's reformulation. The hypothesis $\#A_\mathfrak{p}(\mathbf{F}_\mathfrak{p}) \equiv 0(\ell)$ is equivalent to $\det(I - \overline{\rho_\ell}(\mathrm{Frob}_\mathfrak{p})) = 0$ in $\mathbf{F}_\ell$. Since S has density 1, the Čebotarev Density Theorem implies that $\det(I - \overline{\rho_\ell}(\sigma)) = 0$ in $\mathbf{F}_\ell$ for all $g \in G_K$, *i.e.* every $\overline{\rho_\ell}(g) \in \operatorname{im}\overline{\rho_\ell}$ has a fixed point.

The existence of a K-isogenous abelian variety A' having a global point of order ℓ is equivalent to the existence of G_K-stable lattices $\mathcal{L}' \subset \mathcal{L}$ in $T_\ell(A)\otimes\mathbf{Q}_\ell$, such that $[\mathcal{L} : \mathcal{L}'] = \ell$ and G_K acts trivially on the quotient $\mathcal{L}/\mathcal{L}'$. Thus the semisimplification of the mod ℓ representation $\mathcal{L}'\otimes\mathbf{F}_\ell$ contains the trivial representation. The semisimplification is independent of the lattice $\mathcal{L}'$ by the Brauer-Nesbitt theorem [5, p. 215], hence the semisimplification of $T_\ell(A)\otimes\mathbf{F}_\ell$ contains the trivial representation.

Conversely, if the semisimplification of $T_\ell(A)\otimes\mathbf{F}_\ell = T_\ell(A)/\ell T_\ell(A)$ contains the trivial representation, then for some filtration

$$\ell T_\ell(A) \subset \cdots \subset \mathcal{L}_i \subset \cdots \subset T_\ell(A)$$

of $T_\ell(A)/\ell T_\ell(A)$, there exists an index i such that $\mathcal{L}_i/\mathcal{L}_{i+1}$ is trivial. But $\mathcal{L}_i/\mathcal{L}_{i+1} \subset (\ell^{-1}\mathcal{L}_{i+1})\otimes\mathbf{F}_\ell$. This produces a lattice (and therefore an abelian variety) whose reduction modulo ℓ contains the trivial representation.

With the reformulation complete, we briefly describe Katz's counterexample when $\dim A \geq 3$. Let A_i, $i = 1, 2, 3$, be abelian varieties of dimension d_i defined over a number field K, where K is taken so that $A_i[\ell] \subset A_i(K)$, and let ϵ_1 and ϵ_2 be distinct quadratic characters. We have an embedding:

$$\begin{aligned}\mathbf{Z}/2\times\mathbf{Z}/2 &\hookrightarrow \mathrm{Aut}_{\overline{K}}(A_1\times A_2\times A_3)\\ &(P_1,P_2,P_3)\mapsto(\sigma_1P_1,\sigma_2P_2,\sigma_1\sigma_2P_3),\quad(\sigma_i\in\{\pm1\}).\end{aligned}$$

Use this action to twist $(A_1\times A_2\times A_3)/K$ to get A/K with

$$\overline{\rho_{\ell,A}}(g) = \begin{pmatrix} \epsilon_1(g)I_{d_1} & & \\ & \epsilon_2(g)I_{d_2} & \\ & & \epsilon_1\epsilon_2(g)I_{d_3}\end{pmatrix}.$$

No simple factor of the representation is trivial, yet every element of $\operatorname{im}\overline{\rho_\ell}$ has 1 as an eigenvalue.

3. Background on Symplectic Geometry

The proof of Theorem 1 relies on the subgroup structure of the symplectic group $\mathrm{Sp}_6(\mathbf{F}_\ell)$. In this section we describe the maximal subgroups of $\mathrm{Sp}_6(\mathbf{F}_\ell)$, and outline a general program to find the subgroups of $\mathrm{Sp}_n(\mathbf{F}_\ell)$; for a more detailed account see [10, ch. 2-4]. Recall that the notation $A.B$ is used for any middle term of a short exact sequence of groups with kernel A and quotient B. Following [4, p. xx], this notation is left-associated, so that $A.B.C$ means $(A.B).C$ and has A as a normal subgroup.

The maximal subgroups of $\mathrm{Sp}_6(\mathbf{F}_\ell)$ come in two types: those which stabilize a vector space decomposition of $\mathbf{F}_\ell^6$, and those which do not. We refer to the former as *geometric* subgroups (or Lie subgroups), and to the latter as *exotic* subgroups (type $\mathcal{S}$ in [10]). It is known that when $n \leq 5$ the number of exotic subgroups of $\mathrm{Sp}_{2n}(\mathbf{F}_\ell)$ is bounded independently of ℓ [9, p. 188-219].

We now describe the various vector space decompositions which will give rise to the maximal geometric subgroups of $\mathrm{Sp}_6(\mathbf{F}_\ell)$. Let ℓ be a prime number and write V_{2n} for a symplectic vector space of dimension

$2n$ over $\mathbf{F}_\ell$ with basis $\{e_i, f_i\}_{i=1,\ldots,n}$. Let $\mathrm{Sp}_{2n}(\mathbf{F}_\ell)$ be the subgroup of $\mathrm{GL}_{2n}(\mathbf{F}_\ell)$ preserving the symplectic form $\langle\ ,\ \rangle$, where

$$\langle e_i, e_j\rangle = \langle f_i, f_j\rangle = 0, \text{ and } \langle e_i, f_j\rangle = \delta_{ij}.$$

We denote by $\mathcal{J}_{2n}$ the non-degenerate, skew-symmetric matrix associated to this form.

There are seven decompositions of V_6 which give rise to maximal subgroups of $\mathrm{Sp}_6(\mathbf{F}_\ell)$. We now give brief descriptions of these subgroups; our treatment is based on [10, ch. 2].

3.1. $\mathrm{Sp}_4(\mathbf{F}_\ell) \times \mathrm{SL}_2(\mathbf{F}_\ell)$**.** The subgroup $\mathrm{Sp}_4(\mathbf{F}_\ell) \times \mathrm{SL}_2(\mathbf{F}_\ell)$ of $\mathrm{Sp}_6(\mathbf{F}_\ell)$ is the stabilizer of the vector space decomposition $V_6 = V_4 \oplus V_2$, and is embedded in $\mathrm{Sp}_6(\mathbf{F}_\ell)$ via the symplectic form $\begin{pmatrix} \mathcal{J}_4 & \\ & \mathcal{J}_2 \end{pmatrix}$.

3.2. $\mathrm{SL}_2(\mathbf{F}_\ell) \wr S_3$**.** The decomposition V_6 into subspaces of the same dimension yields two maximal subgroups: $\mathrm{SL}_2(\mathbf{F}_\ell) \wr S_3$ and $\mathrm{GL}_3(\mathbf{F}_\ell).2$. The former stabilizes the decomposition $V_6 = V_2^{\oplus 3}$, and is embedded in $\mathrm{Sp}_6(\mathbf{F}_\ell)$ via the form $\begin{pmatrix} \mathcal{J}_2 & & \\ & \mathcal{J}_2 & \\ & & \mathcal{J}_2 \end{pmatrix}$. The latter stabilizes the decomposition of V_6 into two totally-singular spaces, *e.g.* the subspaces spanned by $\{e_i\}_{i=1}^3$ and $\{f_i\}_{i=1}^3$ respectively. The embedding $\mathrm{GL}_3(\mathbf{F}_\ell).2 \hookrightarrow \mathrm{Sp}_6(\mathbf{F}_\ell)$ is via $g \mapsto \begin{pmatrix} g & \\ & g^* \end{pmatrix}$, where $*$ denotes inverse-transpose, together with an involution which permutes g and g^*.

3.3. **The Field Extension Subgroups.** The field $\mathbf{F}_q$ consisting of $q = \ell^m$ elements is naturally a vector space of dimension m over $\mathbf{F}_\ell$, whence the embedding

$$\mathrm{GL}_n(\mathbf{F}_q) \hookrightarrow \mathrm{GL}_{nm}(\mathbf{F}_\ell).$$

The action of $\mathrm{Gal}(\mathbf{F}_q/\mathbf{F}_\ell) \simeq \mathbf{Z}/m$ on $\mathbf{F}_q$ is compatible with this embedding, resulting in the subgroup $\mathrm{GL}_n(\mathbf{F}_q).m$ of $\mathrm{GL}_{nm}(\mathbf{F}_\ell)$. In the case of $\mathrm{Sp}_6(\mathbf{F}_\ell)$, the subgroups $\mathrm{SL}_2(\mathbf{F}_{\ell^3}).3$ and $\mathrm{GU}_3(\mathbf{F}_{\ell^2}).2$ obtained in this way are maximal [10, lem. 4.3.7, 4.3.10].

3.4. $\mathrm{O}_3(\mathbf{F}_\ell) \otimes \mathrm{SL}_2(\mathbf{F}_\ell)$**.** The vector space decomposition $V_6 = E_3 \otimes V_2$, where E_3 is a three-dimensional vector space equipped with a symmetric form, yields the maximal subgroup $\mathrm{O}_3(\mathbf{F}_\ell) \otimes \mathrm{SL}_2(\mathbf{F}_\ell)$ of $\mathrm{Sp}_6(\mathbf{F}_\ell)$. This is the image of the tensor product representation of $\mathrm{O}_3(\mathbf{F}_\ell) \times \mathrm{SL}_2(\mathbf{F}_\ell)$, where $(g,h)\cdot v \otimes w = gv \otimes hw$.

3.5. **Parabolic Subgroups.** A *parabolic* subgroup G of $\mathrm{Sp}_6(\mathbf{F}_\ell)$ is the stabilizer of a flag $\mathcal{F}$ of totally singular subspaces

$$0 \subset W_1 \subset \cdots \subset W_k$$

(so W_k is at most three-dimensional). Since any totally singular subspace W is contained in $W^\perp$, a flag $\mathcal{F}$ gives rise to a chain of subspaces

$$0 \subset W_1 \subset \cdots \subset W_k \subset W_k^\perp \subset \cdots \subset W_1^\perp \subset V_6.$$

By Witt's Lemma [10, p. 18], any flag is conjugate to one in which the W_i are spanned by standard basis vectors. When the length k of $\mathcal{F}$ is maximal, it's stabilizer is (conjugate to) the upper-triangular subgroup of $\mathrm{Sp}_6(\mathbf{F}_\ell)$. The stabilizer of any subflag $\mathcal{F}$ is a block upper-triangular subgroup of $\mathrm{Sp}_6(\mathbf{F}_\ell)$.

3.6. **Exotic Subgroups.** The exotic subgroups of the finite simple groups of Lie type of dimension ≤ 11 are classified in [9, pp. 188-219]; for $\mathrm{PSp}_6(\mathbf{F}_\ell)$, $\ell \geq 7$ they are given in the following table:

Group	Conditions
$\mathrm{PSL}_2(\mathbf{F}_\ell)$	$\ell \geq 7$
S_5	$\ell \equiv \pm 1(8)$
A_5	$\ell \equiv \pm 3(8)$
$\mathrm{PSL}_2(\mathbf{F}_7).a$	$\ell \notin \{2,3,7\}$, $\left(\frac{13}{\ell}\right) = 1$ $a = 2$ if $q \equiv \pm 1(16)$ $a = 1$ if $q \equiv \pm 3, \pm 5, \pm 7(16)$
$\mathrm{PSL}_2(\mathbf{F}_{13})$	$\ell \notin \{2,13\}$, $\left(\frac{13}{\ell}\right) = 1$
$\mathrm{PSU}_3(\mathbf{F}_9)$	$\ell \equiv \pm 1(12)$
J_2	$\left(\frac{5}{\ell}\right) = 1$

In addition, the groups $\mathrm{Sp}_4(\mathbf{F}_\ell)$ and $\mathrm{GU}_3(\mathbf{F}_{\ell^2})$ (which are geometric subgroups of $\mathrm{Sp}_6(\mathbf{F}_\ell)$) contain the *symplectic-type normalizer* subgroups $2.2^4.\mathrm{O}_4^-(\mathbf{F}_2)$ and $3.3^2.\mathrm{SL}_2(\mathbf{F}_3)$, respectively [10, ch. 4.6]. Here the group $\mathbf{Z}/2$ (resp. $\mathbf{Z}/3$) in front is the center of $2.2^4.\mathrm{O}_4^-(\mathbf{F}_2)$ (resp. $3.3^2.\mathrm{SL}_2(\mathbf{F}_3)$). The group $\mathrm{O}_4^-(\mathbf{F}_2)$ (resp. $\mathrm{SL}_2(\mathbf{F}_3)$) acts on $2^4 \simeq \mathbf{F}_2^4$ (resp. $3^2 \simeq \mathbf{F}_3^2$) by conjugation.

We finish this section by recalling two theorems of basic group theory which are used extensively in the text.

Clifford's Theorem [5, p. 343]. *Let M be an irreducible KG-module where K is an arbitrary field, and let $H \lhd G$. Then M_H is a completely reducible KH-module, and the irreducible KH-submodules of M_H are all conjugates of each other.*

Goursat's Lemma [3, p. 864]. *Let A and B be finite groups. The subgroups G of $A \times B$ are in one-to-one correspondence with the tuples (G_1, G_2, G_3, ψ) where $G_1 \leq A$, $G_2 \leq B$, $G_3 \lhd G_2$, and $\psi : G_1 \longrightarrow G_2/G_3$ is a surjective homomorphism.*

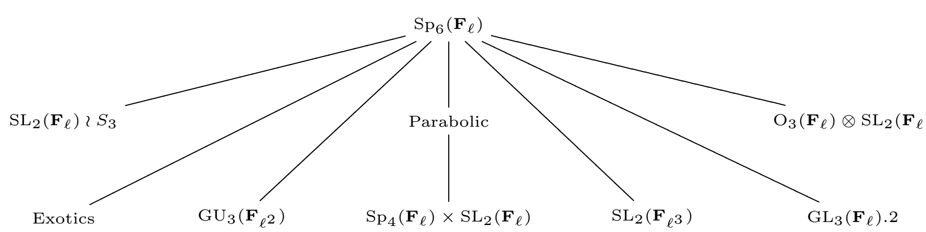
We set the following terminology for the rest of the paper: call the quadruple (G_1, G_2, G_3, ψ) the *Goursat-tuple* associated to $G \subset A \times B$. If G is not a direct product, then we call G a *Goursat-subgroup* of $A \times B$, and write $G = G_1 \bullet G_2$. If a linear representation of a group G has the property that every $g \in G$ has 1 as an eigenvalue (the *fixed-point assumption*), yet the semisimplification of the action of G on its natural module does not contain the trivial representation, we call G an *obstruction* to the local-global principle.

4. Strategy of Proof

Katz's reformulation (Question 2) implies that if A/K is an abelian variety for which Lang's question (Question 1) has a negative answer, then $\mathrm{im}\,\overline{\rho_\ell} = G \subset \mathrm{GSp}_6(\mathbf{F}_\ell)$ has the property that $\det(I - g) = 0$ for all $g \in G$, yet the semisimplification of the action of G on $T_\ell(A) \otimes \mathbf{F}_\ell$ does not contain the trivial representation. Conversely, if G is any such subgroup of $\mathrm{GSp}_6(\mathbf{F}_\ell)$, then G is realizable as $\mathrm{im}\,\overline{\rho_\ell}$ for some abelian threefold over a number field. Indeed, if $A/\mathbf{Q}$ is any threefold such that $\mathrm{im}\,\overline{\rho_\ell} = \mathrm{GSp}_6(\mathbf{F}_\ell)$, then the base field extension from $\mathbf{Q}$ to $K := \overline{\mathbf{Q}}^G$ produces a threefold over a number field which is a counterexample to Question 1. Therefore, to classify all obstructions to the local-global principle for ℓ-torsion on abelian threefolds, it suffices to enumerate all subgroups G of $\mathrm{GSp}_6(\mathbf{F}_\ell)$ for which $\det(I - g) = 0$ for all $g \in G$, and whose semisimplification does *not* contain the trivial representation.

The two assumptions on $\overline{\rho_\ell}$ in Theorem 1 are that $\mathrm{im}\,\overline{\rho_\ell} \subset \mathrm{Sp}_6(\mathbf{F}_\ell)$ and $\ell \geq 7$. The first allows us to capture the essence of the problem (symplectic geometric algebra in dimension 6), while simplifying some of the computations (the determinant is 1). The first assumption is equivalent to K containing all ℓ^{th} roots of unity. The second assumption is made so that $\mathrm{char}(\mathbf{F}_\ell)$ is coprime to $\#\,\mathrm{im}\,\overline{\rho_\ell}$ in the cases where $\mathrm{im}\,\overline{\rho_\ell}$ is of Katz type or Exceptional type. For the Lie type counterexamples almost no modular representation theory (where $\ell | \# G$) is needed.

To carry out the program, we start with the maximal subgroups of $\mathrm{Sp}_6(\mathbf{F}_\ell)$:



Each of these maximal subgroups turns out to be too large for every element to have a fixed point (each contains $-I$, for example), so any obstruction is necessarily contained in some maximal subgroup of one of these subgroups. Using the classification in Section 1.3 we see that these new maximal subgroups are still too large for the fixed-point condition to hold, so we need to go to another level in the lattice. These "level 2" maximal subgroups are also of Lie-type so we can iterate this procedure, and all obstructions will eventually be found. Using techniques of geometric algebra, representation theory, and finite group theory we are able to solve this problem without going too far into the maximal subgroup lattice of $\mathrm{Sp}_6(\mathbf{F}_\ell)$.

The Katz-type obstructions occur near the bottom of the subgroup lattice; in fact, each maximal subgroup of $\mathrm{Sp}_6(\mathbf{F}_\ell)$ contains a Katz-type obstruction. The other obstructions occur higher up in the lattice and exhibit interesting properties: the orders of the Lie-type obstructions grow with ℓ, and the Exceptional-type obstructions give rise to abelian varieties with special properties.

We end this section with a simplifying remark. Any parabolic subgroup of $\mathrm{Sp}_6(\mathbf{F}_\ell)$ is a semidirect product of its Levi subgroup and its unipotent radical. Moreover, the Levi subgroup is precisely the semisimplification of G. It is easy to check using the symplectic form on $\mathbf{F}_\ell^6$ that the Levi subgroup is a subgroup of one of the geometric subgroups $\mathrm{Sp}_4(\mathbf{F}_\ell) \times \mathrm{SL}_2(\mathbf{F}_\ell)$, $\mathrm{GL}_3(\mathbf{F}_\ell)$, or $\mathrm{SL}_2(\mathbf{F}_\ell) \times \mathrm{SL}_2(\mathbf{F}_\ell) \times \mathrm{SL}_2(\mathbf{F}_\ell)$.

Since we are only concerned with $T_\ell(A) \otimes \mathbf{F}_\ell$ up to semisimplification, it suffices to work with semisimple representations. Therefore, without loss of generality, we can assume that $\operatorname{im} \overline{\rho_\ell}$ is not parabolic. We will sometimes refer to the semisimplification of a $K[G]$-module M as the *semisimplification of* G, when the action of $K[G]$ on M is clear. This terminology is not standard.

5. Proof of the main theorem: subgroups of $\mathrm{Sp}_4(\mathbf{F}_\ell) \times \mathrm{SL}_2(\mathbf{F}_\ell)$

We divide the proof of Theorem 1 over the next five sections, based on the maximal subgroup of $\mathrm{Sp}_6(\mathbf{F}_\ell)$ containing $\operatorname{im} \overline{\rho_\ell}$. It is easy to exhibit in each maximal geometric subgroup of $\mathrm{Sp}_6(\mathbf{F}_\ell)$ elements without 1 as an eigenvalue (*e.g.* $-I$), so $\operatorname{im} \overline{\rho_\ell}$ is necessarily a proper subgroup of one of these maximal subgroups. In this section we work with the maximal subgroup $\mathrm{Sp}_4(\mathbf{F}_\ell) \times \mathrm{SL}_2(\mathbf{F}_\ell)$. Let $G \subset \mathrm{Sp}_4(\mathbf{F}_\ell) \times \mathrm{SL}_2(\mathbf{F}_\ell)$ be an obstruction corresponding to a Goursat-tuple (G_1, G_2, G_3, ψ), so that $G_1 \subset \mathrm{Sp}_4(\mathbf{F}_\ell)$, $G_2 \subset \mathrm{SL}_2(\mathbf{F}_\ell)$, $G_3 \lhd G_2$, and $\psi : G_1 \twoheadrightarrow G_2/G_3$ is a homomorphism.

We begin by deriving some general properties of G which will be used extensively throughout the paper. The remainder of this section is then divided according to the maximal subgroup of $\mathrm{Sp}_4(\mathbf{F}_\ell)$ containing G_1. The maximal geometric subgroups of $\mathrm{Sp}_4(\mathbf{F}_\ell)$ are [10, p. 72]:

$$\mathrm{GL}_2(\mathbf{F}_\ell).2,\ \mathrm{SL}_2(\mathbf{F}_\ell) \wr S_2,\ \mathrm{SL}_2(\mathbf{F}_{\ell^2}).2,\ \mathrm{GU}_2(\mathbf{F}_{\ell^2}).2,\ \text{and } 2.2^4.\mathrm{O}_4^-(\mathbf{F}_2),$$

while the exotic subgroups of $\mathrm{PSp}_4(\mathbf{F}_\ell)$ are $\mathrm{PSL}_2(\mathbf{F}_\ell)$, S_6, and A_6 [9, p. 209]. The preimage in $\mathrm{Sp}_4(\mathbf{F}_\ell)$ of any exotic subgroup of $\mathrm{PSp}_4(\mathbf{F}_\ell)$ is a central extension of degree at most 2.

With the exception of $2.2^4.\mathrm{O}_4^-(\mathbf{F}_2)$, the geometric subgroups of $\mathrm{Sp}_4(\mathbf{F}_\ell)$ are analogous to those of $\mathrm{Sp}_6(\mathbf{F}_\ell)$. In this section, we will not consider the case where $G_1 \subset \mathrm{SL}_2(\mathbf{F}_\ell) \wr S_2$, since it is subsumed by the next section on $\mathrm{SL}_2(\mathbf{F}_\ell) \wr S_3$.

Before we proceed with the case-by-case analysis of the subgroups of $\mathrm{Sp}_4(\mathbf{F}_\ell) \times \mathrm{SL}_2(\mathbf{F}_\ell)$, we summarize the results of this section in the following table.

Subgroup of $\mathrm{Sp}_4(\mathbf{F}_\ell) \times \mathrm{SL}_2(\mathbf{F}_\ell)$	Obstruction	Reference
$\mathrm{SL}_2(\mathbf{F}_{\ell^2}).2 \times \mathrm{SL}_2(\mathbf{F}_\ell)$	None	Proposition 1
$\mathrm{GU}_2(\mathbf{F}_{\ell^2}).2 \times \mathrm{SL}_2(\mathbf{F}_\ell)$	$\mathbf{Z}/2 \times \mathbf{Z}/2$ D_4, D_n $\mathrm{GL}_2(\mathbf{F}_3)$	Proposition 2
	$(\mathbf{Z}/2 \times \mathbf{Z}/2).2$ $D_4.2$, $D_n.2$ $\mathrm{GL}_2(\mathbf{F}_3).2$	Lemmas 6 and 7
$\mathrm{GL}_2(\mathbf{F}_\ell).2 \times \mathrm{SL}_2(\mathbf{F}_\ell)$	$\mathbf{Z}/2 \times \mathbf{Z}/2$, D_4, $D_4.2$, $\mathrm{GL}_2(\mathbf{F}_3)$, $\mathrm{GL}_2(\mathbf{F}_3).2$ D_n, $D_n.2$	Proposition 4
$2.2^4.\mathrm{O}_4^-(\mathbf{F}_2) \times \mathrm{SL}_2(\mathbf{F}_\ell)$	None	Lemmas 8 and 9
$2.S_6 \times \mathrm{SL}_2(\mathbf{F}_\ell)$	$\mathbf{Z}/2 \times \mathbf{Z}/2$, S_3 $2.(\mathbf{Z}/2 \times \mathbf{Z}/2)$, $2.S_3$	Proposition 5
$\mathrm{Sym}^3(\mathrm{SL}_2(\mathbf{F}_\ell)) \times \mathrm{SL}_2(\mathbf{F}_\ell)$	None	Proposition 6

5.1. Basic Properties of Obstructions in $\mathrm{Sp}_4(\mathbf{F}_\ell) \times \mathrm{SL}_2(\mathbf{F}_\ell)$.

Lemma 1. *Let $G \subset \mathrm{Sp}_4(\mathbf{F}_\ell) \times \mathrm{SL}_2(\mathbf{F}_\ell)$ be an obstruction given by the Goursat-tuple (G_1, G_2, G_3, ψ). Then* *(a) G is a Goursat-subgroup, and*

(b) G_3 is either trivial or, when G_2 is Borel, a subgroup of the Sylow-ℓ subgroup of G_2.

Proof. Katz's result on abelian varieties in dimensions 1 and 2 [8, p. 492] implies that any subgroup of $\mathrm{Sp}_4(\mathbf{F}_\ell)$ or $\mathrm{SL}_2(\mathbf{F}_\ell)$ satisfying the fixed-point assumption necessarily has a copy of the trivial representation in its semisimplification. If G is a direct product and satisfies the fixed-point assumption, then so does G_1 or G_2, hence either G_1 or G_2 each have a copy of the trivial representation in their semisimplifications. The semisimplification of the action of G on symplectic 6-space is comprised of that of G_1 on symplectic 4-space and G_2 on symplectic 2-space (since G is embedded in $\mathrm{Sp}_6(\mathbf{F}_\ell)$ in block-diagonal form), so G contains a copy of the trivial representation in its semisimplification. Therefore G must be a Goursat-subgroup of $\mathrm{Sp}_4(\mathbf{F}_\ell) \times \mathrm{SL}_2(\mathbf{F}_\ell)$.

For (b) we know there exists some $g_1 \in G_1$ without 1 as an eigenvalue, in light of Katz's result for abelian surfaces. The coset $\psi(g_1) = g_2 G_3$ must consist entirely of elements having at least one eigenvalue 1, and hence both equal to 1 since $G_2 \subset \mathrm{SL}_2(\mathbf{F}_\ell)$. To see that G_3 consists entirely of eigenvalue-1 elements, first observe that g_2 has both eigenvalues equal to 1 since G_3 contains the identity of $\mathrm{SL}_2(\mathbf{F}_\ell)$. Let $g \in G_3$ and pick a basis for $\mathbf{F}_\ell^2$ so that g_2 is upper-triangular. The following computation reveals that g has trace 2:

$$\mathrm{tr}(g_2 g) = \mathrm{tr}\left[\left(\begin{smallmatrix} 1 & e \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right] = a + d + ce = 2$$

$$\mathrm{tr}(g_2 g^{-1}) = \mathrm{tr}\left[\left(\begin{smallmatrix} 1 & e \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)\right] = a + d - ce = 2$$

The trace and determinant are independent of basis, so every $g \in G_3$ has both eigenvalues equal to 1, as desired. ☐

Remark. If G_2 is a Borel subgroup of $\mathrm{SL}_2(\mathbf{F}_\ell)$, then G_3 is either trivial or cyclic of order ℓ. We will *always* assume G_3 is cyclic of order ℓ. This assumption actually *weakens* inequality 1 below and will afford us the greater flexibility to find all the obstructions. It will become clear that no obstruction can exist when G_2 is Borel and G_3 is trivial.

Corollary 1. *If $g_1 \in G_1$ does not have 1 as an eigenvalue, then $g_1 \in \ker \psi$.*

Proof. Since $\psi(g_1) = g_2 G_3$ consists entirely of elements having both eigenvalues equal to 1, Lemma 1(b) implies $g_2 \in G_3$. ☐

In light of Lemma 1 it makes sense to define a subset P of G_1 by

$$P = \{g_1 \in G_1 \ : \ g_1 \text{ does not have a fixed point}\},$$

and by Corollary 1, we have $\# \ker \psi \geq \# P$. Since $\psi : G_1 \longrightarrow G_2/G_3$ is a surjective homomorphism, the following inequality must be satisfied whenever G is an obstruction:

$$\frac{\# G_1}{[G_2 : G_3]} \geq \# P + 1, \tag{1}$$

where the extra +1 comes from the identity of G_1. In order to apply this inequality to certain subgroups of $\mathrm{Sp}_6(\mathbf{F}_\ell)$, we make the following observation, whose proof is elementary.

Lemma 2. *The only triples $(n, m, k) \in \mathbf{Z}^3_{\geq 2}$ satisfying $nm/k \geq (n-1)(m-1)+1$ are $(2, m, 2)$ and $(n, 2, 2)$.*

Lemma 3. *Let K be a finite field of characteristic $\neq 2$ and let $G = \begin{pmatrix} G_1 & & \\ & G_2 & \\ & & G_3 \end{pmatrix} \subset \mathrm{GL}_3(K)$ be a diagonal subgroup (so the G_i are cyclic subgroups of $K^\times$). If every $g \in G$ has 1 as an eigenvalue, and the semisimplification of G does not contain the trivial representation, then $G \simeq \mathbf{Z}/2 \times \mathbf{Z}/2$.*

Proof. Using Goursat's Lemma twice, we can write $G = (G_1 \bullet G_2) \bullet G_3$ (where $\bullet$ is allowed to denote a direct product for this proof only), with Goursat-tuples

$$(G_1 \bullet G_2, G_3, H_3, \psi) \text{ and } (G_1, G_2, H_2, \phi)$$

associated to $(G_1 \bullet G_2) \bullet G_3$ and $G_1 \bullet G_2$ respectively. Denote the orders of G_1, H_2, and $[G_3 : H_3]$ by n, m, and k respectively.

By [12, p. I-2, ex. 1], there exists $g \in G_1 \bullet G_2$ without1 as an eigenvalue. Hence $\psi(g)$ must be trivial, which means H_3 must have size 1 (*i.e.* is trivial).

There are $\#\ker\phi + \#H_2 - 1 = \#\ker\phi + m - 1$ elements of $G_1 \bullet G_2$ with 1 as an eigenvalue (the identity was counted twice). In this case, inequality (1) becomes

$$\frac{nm}{k} \geq \underbrace{nm - \#\ker\phi - m + 1}_{\#P} + 1. \tag{2}$$

In the extreme case where $\#\ker\phi = n$ we have:

$$\frac{nm}{k} \geq \underbrace{nm - n - m + 1}_{\#P} + 1 = (n-1)(m-1) + 1. \tag{3}$$

It suffices to assume $n, m, k \geq 2$. Lemma 2 applies to this case, and without loss of generality we will assume $n = k = 2$ and m is arbitrary. The algebraic conditions imposed by ϕ and ψ force $m = 2$. Indeed, we can violate (3) by exhibiting one additional element of $\ker\psi$ whenever $m > 2$: if $(-1, b) \in \ker\psi$, then so is $(-1, b)^2 = (1, b^2)$. If b^2 is non-trivial, this gives an additional element of the kernel (if b^2 is trivial, then $\#G_2 = 2$), contradicting (3).

In general, set $\#\ker\phi = n_0$, for some non-trivial proper divisor n_0 of n. We therefore have:

$$\frac{nm}{k} \geq nm - n_0 - m + 1 + 1 = (n-1)(m-1) + 1 + (n - n_0)\,.$$

Since $(n - n_0) > 0$, there are no solutions $(n, m, k) \in \mathbf{Z}^3_{\geq 2}$ to this inequality. It is not hard to check that the hypotheses on G are satisfied if and only if $G = \begin{pmatrix} \epsilon_1 & & \\ & \epsilon_2 & \\ & & \epsilon_1\epsilon_2 \end{pmatrix}$, where $\epsilon_i \in \{\pm 1\}$, which proves the lemma. □

Remark. The embedding $\mathbf{Z}/2 \times \mathbf{Z}/2 \simeq G \hookrightarrow \mathrm{Sp}_6(\mathbf{F}_\ell)$ as 2×2 block-diagonal matrices is precisely Katz's counterexample for abelian threefolds.

Lemma 4. *Let $G \subset \mathrm{Sp}_4(\mathbf{F}_\ell) \times \mathrm{SL}_2(\mathbf{F}_\ell)$ be an obstruction with Goursat-tuple (G_1, G_2, G_3, ψ). Then the following conditions hold for G:*

*(a) there exist elements of G_1 without a fixed point (*i.e. *none of the eigenvalues is 1), and*

(b) there are non-identity elements of G_1 having a fixed point.

Proof. The proof of this lemma relies on the fact that $\mathrm{Sp}_4(\mathbf{F}_\ell) \times \mathrm{SL}_2(\mathbf{F}_\ell)$ embeds in $\mathrm{Sp}_6(\mathbf{F}_\ell)$ in block form:

$$\mathrm{Sp}_4(\mathbf{F}_\ell) \times \mathrm{SL}_2(\mathbf{F}_\ell) \hookrightarrow \begin{pmatrix} * & * & * & * & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \end{pmatrix}.$$

The uniqueness of the simple factors of the Jordan-Hölder series for $\mathbf{F}_\ell^6$ as an $\mathbf{F}_\ell[G]$-module implies that the semisimplification of G is comprised of the semisimplifications of G_1 and G_2 respectively.

Part (a) follows from Katz's result for abelian surfaces [8, p. 492]. For (b), observe that if the only element of G_1 with 1 as an eigenvalue were the identity, then $\ker\psi = G_1$, contradicting the assumption that G be an obstruction. □

5.2. **Case 1:** $G_1 \subset \mathrm{SL}_2(\mathbf{F}_{\ell^2}).2$**.** In this section we assume $G_1 \subset \mathrm{SL}_2(\mathbf{F}_{\ell^2}).2$, where $\mathrm{SL}_2(\mathbf{F}_{\ell^2}).2$ is a maximal field extension subgroup of $\mathrm{Sp}_4(\mathbf{F}_\ell)$ [10, (4.3.11)]. The following lemma compares the eigenvalues of $\mathrm{SL}_2(\mathbf{F}_{\ell^2})$ acting on $\mathbf{F}_{\ell^2}^2$ to the eigenvalues on $\mathbf{F}_\ell^4$ via its embedding into $\mathrm{Sp}_4(\mathbf{F}_\ell)$.

Lemma 5. *Let $\iota : \mathrm{SL}_2(\mathbf{F}_{\ell^2}).2 \hookrightarrow \mathrm{Sp}_4(\mathbf{F}_\ell)$ be the field-extension embedding. Then $g \in \mathrm{SL}_2(\mathbf{F}_{\ell^2})$ has 1 as an eigenvalue if and only if $\iota(g)$ has 1 as an eigenvalue.*

Proof. An element g of $\mathrm{SL}_2(\mathbf{F}_{\ell^2})$ has 1 as an eigenvalue if and only if it is conjugate (by $S \in \mathrm{SL}_2(\mathbf{F}_\ell)$, say) to $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$. Apply the field-extension embedding to $S^{-1}gS$ and we are done. □

Corollary 2. *If $G_1 \subset \mathrm{SL}_2(\mathbf{F}_{\ell^2})$, then G cannot be an obstruction.*

Proof. Suppose $\#G_1 = \ell^e k$, where $e \in \{0,1,2\}$ and $(k,\ell) = 1$. According to inequality (1),

$$\begin{aligned}\frac{\#G_1}{[G_2 : G_3]} &\geq \#P + 1 \\ &= \underbrace{\#G_1 - \#\{\iota(g) \in \iota(G_1) \ : \ \iota(g) \text{ has a fixed point}\}}_{\#P} + 1 \\ &= \#G_1 - \#\{g \in G_1 \ : \ g \text{ has a fixed point}\} + 1 \text{ (by Lemma 5).}\end{aligned}$$

If $e = 0$, then the inequality becomes:

$$\frac{k}{[G_2 : G_3]} \geq k - 1 + 1,$$

which has a solution if and only if $G_2 = G_3$, *i.e.* if an only if G is a direct product, contradicting Lemma 1(a).

If $e \in \{1,2\}$, then G_1 has at least ℓ elements of order ℓ, and therefore having both eigenvalues equal to 1. In this case the inequality is

$$\frac{k\ell^e}{[G_2 : G_3]} \geq k\ell^e - \ell + 1,$$

which forces $G_2 = G_3$ again. Therefore G cannot be an obstruction. □

An arbitrary subgroup G of $\mathrm{SL}_2(\mathbf{F}_{\ell^2}).2 \times \mathrm{SL}_2(\mathbf{F}_\ell)$ gives rise to a Goursat-tuple (G_1, G_2, G_3, ψ). We define the subgroup G^0 of G as follows. There is an exact sequence [10, (4.3.11)]

$$1 \longrightarrow \mathrm{SL}_2(\mathbf{F}_{\ell^2}) \longrightarrow \mathrm{SL}_2(\mathbf{F}_{\ell^2}).2 \xrightarrow{\pi} \mathbf{Z}/2 \longrightarrow 1,$$

hence there exists an index-2 subgroup $G_1^0 := G_1 \cap \ker \pi$ of G_1, so that $G_1^0 \subset \mathrm{SL}_2(\mathbf{F}_{\ell^2})$. If we set $\psi^0 := \psi|_{G_1^0}$, and define G_2^0 and G_3^0 accordingly (so that $G_2^0/G_3^0 = \psi^0(G_1^0)$ and $G_3^0 = \psi^0(\ker \psi^0)$), then G^0 is the subgroup of $\mathrm{SL}_2(\mathbf{F}_{\ell^2}) \times \mathrm{SL}_2(\mathbf{F}_\ell)$ with Goursat-tuple $(G_1^0, G_2^0, G_3^0, \psi^0)$.

Proposition 1. *No subgroup $G \subset \mathrm{SL}_2(\mathbf{F}_{\ell^2}).2 \times \mathrm{SL}_2(\mathbf{F}_\ell)$ can be an obstruction.*

Proof. By assumption, G^0 satisfies the fixed-point condition and by Corollary 2, the semisimplification of G^0 contains the trivial representation. Thus, either the semisimplification of G_1^0 or G_2^0 contains the trivial representation.

If the semisimplification of G_2^0 contains the trivial representation, then $G_2^0 = G_3^0$ and G_2^0 is either trivial or has order ℓ. Since $\#\ker \psi = \#G_1/2$, it follows that $G_2/G_3 \simeq \{\pm I\}$, which implies *every* element of $G_1 - G_1^0$ must have 1 as an eigenvalue. However, $g \in G_1^0$ has 1 as an eigenvalue if and only if g^σ does [10, (2.1.2)]. Therefore, every element of G_1 has 1 as an eigenvalue and so the semisimplification of G contains the trivial representation.

It remains to consider the case where the trivial representation occurs in the semisimplification of G_1^0. If so, $\#G_1^0|\ell^2$, $[G_1 : G_1^0] = 2$, and $G_1 - G_1^0$ cannot contain any element with 1 as an eigenvalue. In order for the fixed-point condition to hold for all of G, it must be the case that $G_2 = G_3$. But this means that G_2 (and hence G) contains the trivial representation in its semisimplification. This proves the proposition. □

5.3. **Case 2:** $G_1 \subset \mathrm{GU}_2(\mathbf{F}_{\ell^2}).2$. We now examine the case where $G_1 \subset \mathrm{GU}_2(\mathbf{F}_{\ell^2}).2$, the other field extension subgroup of $\mathrm{Sp}_4(\mathbf{F}_\ell)$. The subgroups of $\mathrm{GU}_2(\mathbf{F}_{\ell^2}).2$ are the Borel and Cartan (and its normalizer) subgroup, along with the exotic subgroups $2.A_4$, $2.S_4$, and $2.A_5$ (the determinant of the degree-2 representations of these exotic groups is ± 1, hence the representations are unitary [10, (2.3.1)]).

For any subgroup $G_1 \subset \mathrm{GU}_2(\mathbf{F}_{\ell^2}).2$, define $G_1^0 := \ker \pi \cap G_1$, relative to the split exact sequence

$$1 \longrightarrow \mathrm{GU}_2(\mathbf{F}_{\ell^2}) \longrightarrow \mathrm{GU}_2(\mathbf{F}_{\ell^2}).2 \xrightarrow{\pi} \mathbf{Z}/2 \longrightarrow 1.$$

We will first find the obstructions G for which $G_1^0 = G_1$.

Proposition 2. *Let $G_1^0 = G_1 \subset \mathrm{GU}_2(\mathbf{F}_{\ell^2})$. Then G is an obstruction precisely when:*

(a) G is Katz's $\mathbf{Z}/2 \times \mathbf{Z}/2$-counterexample, or
(b) $G_1 \simeq D_4$ or $G_1 \simeq D_n$ ($n|(\ell-1)$) and $G_2/G_3 = \{\pm I\}$, or
(c) $G_1 \simeq 2.S_4 \simeq \mathrm{GL}_2(\mathbf{F}_3)$, and $G_2/G_3 = \{\pm I\}$.

Proof. We use the maximal subgroup structure of $\mathrm{GU}_2(\mathbf{F}_{\ell^2})$. If G_1 is a non-split Cartan subgroup of $\mathrm{GU}_2(\mathbf{F}_{\ell^2})$, then G_1 is cyclic. If a generator for G_1 has 1 as an eigenvalue then we contradict Lemma 4(a). On the other hand, if a generator for G_1 does not have 1 as an eigenvalue, then it is in $\ker\psi$, which implies ψ is the trivial homomorphism and G cannot be an obstruction.

If G_1 is a Borel or a split Cartan subgroup of $\mathrm{GU}_2(\mathbf{F}_{\ell^2})$, then the maximal Cartan subgroup of G_1 is a subgroup of a direct product of (non-trivial) cyclic groups, say H_1 and H_2. Let the maximal Cartan subgroup of G_1 be given by the Goursat-tuple (H_1, H_2, H_3, ϕ), and set $\#H_1 = n, \#H_3 = m$ so that $\#G_1 = nm\ell^e$, where $e \in \{0, 1, 2\}$, and $(nm, \ell) = 1$. Furthermore, G_2 is a subgroup of $\mathrm{SL}_2(\mathbf{F}_\ell)$, so write $\#G_2 = k\ell^f$, where $f \in \{0, 1\}$, and $(k, \ell) = 1$. In this case inequality (1) becomes:

$$\begin{aligned}\frac{nm\ell^e}{k\ell^f/\ell^f} &\geq \underbrace{nm\ell^e - \#\ker\phi - m + 1}_{\#P} + 1\\ &\geq nm\ell^e - (\#\ker\phi)\ell^e - m\ell^e + 2\end{aligned}$$

By the proof of Lemma 3, the only integral triple which satisfies this inequality is $(2, 2, 2)$ (so $\ker\phi = H$), giving us part (a) of the proposition.

Next suppose that G_1 normalizes a split Cartan subgroup C of $\mathrm{GU}_2(\mathbf{F}_{\ell^2})$, where C is given by the Goursat-tuple (H_1, H_2, H_3, ϕ). This means $\#G_1 = 2mn$ and without loss of generality take $m \leq n$. In this case inequality (1) becomes:

$$\begin{aligned}\frac{2nm}{k} &\geq \#P + 1\\ &= \underbrace{nm - \#\ker\phi - \#H_3 + 1}_{\#g\in C \text{ w/o a fixed point}} + \underbrace{nm - \#\{(a, b) \in C \ : \ ab = 1\}}_{\#g\in G_1\setminus C \text{ w/o a fixed point}} + 1\\ &= 2nm - \#\ker\phi - \#H_3 - \#\{(a, b) \in C \ : \ ab = 1\} + 2\\ &\geq 2nm - n - m - m + 2\\ &= 2(n - 1)(m - 1) + n.\end{aligned}$$

If $n, m \geq 2$, then the only triple (n, m, k) satisfying the inequality is $(2, 2, 2)$, which means $G \simeq D_4$. However, if $m = 1$ so that H_3 is trivial and $G_1 = \{(a, \phi(a)) \ : \ a \in H_1\}$, we get a new obstruction. Returning to the inequality, we have

$$\frac{2n}{k} \geq \underbrace{n - \#\ker\phi}_{\#g\in C \text{ w/o a fixed point}} + \underbrace{n - \#\{a \in H_1 \ : \ \phi(a) = a^{-1}\}}_{\#g\in G_1\setminus C \text{ w/o a fixed point}} + 1.$$

Since ϕ is a homomorphism, $\#\{a \in H_1 \ : \ \phi(a) = a^{-1}\} = 1$ or n. When $\#\{a \in H_1 \ : \ \phi(a) = a^{-1}\} = 1$, the inequality is

$$\frac{2n}{k} \geq 2n - \#\ker\phi,$$

where $k \geq 2$ and $\#\ker\phi < n$; both conditions imply the semisimplification of G contains the trivial representation. On the other hand, if $\#\{a \in H_1 \ : \ \phi(a) = a^{-1}\} = n$, then $\ker\phi$ is trivial, giving us the inequality

$$\frac{2n}{k} \geq n.$$

Since $k \geq 2$, we must have $k = 2$. The resulting group G is an obstruction isomorphic to the dihedral group D_n, giving us Part (b) of the proposition. When G_1 is a non-split Cartan subgroup, the inequality becomes

$$\frac{2n}{k} \geq \underbrace{n - 1}_{\#g\in C \text{ w/o a fixed point}} + \#\{g \in G_1 \setminus C \text{ w/o a fixed point}\} + 1.$$

If G is to be an obstruction, *every* element of $G_1 - C$ must have 1 as an eigenvalue, which is impossible.

Finally, let G_1 be an exotic subgroup of $\mathrm{GU}_2(\mathbf{F}_{\ell^2})$. According to Appendix A, the degree-2 representations of $2.A_4$ and $2.A_5$ fail the requirements of Lemma 4(b). Therefore, we can assume $G_1 \simeq 2.S_4 \simeq \mathrm{GL}_2(\mathbf{F}_3)$. Here, G_1 has 16 elements without 1 as eigenvalue, and they all occur in the subgroup $\mathrm{SL}_2(\mathbf{F}_3)$. That means

$\#\ker\psi \geq 17$, whence $\ker\psi = \mathrm{SL}_2(\mathbf{F}_3)$ or $\ker\psi = \mathrm{GL}_2(\mathbf{F}_3)$. If $\ker\psi = \mathrm{GL}_2(\mathbf{F}_3)$, then G_2 contains the trivial representation and we do not get an obstruction, while $\ker\psi = \mathrm{SL}_2(\mathbf{F}_3)$ gives us Part (c) of the proposition. □

When G_1 is an arbitrary subgroup of $\mathrm{GU}_2(\mathbf{F}_{\ell^2}).2$, define the index-2 subgroup G^0 of G as in the paragraph before Proposition 1. If $G \subset \mathrm{GU}_2(\mathbf{F}_{\ell^2}).2 \times \mathrm{SL}_2(\mathbf{F}_\ell)$ is an obstruction, then there are two possibilities: either G^0 is an obstruction, or it is not.

Lemma 6. *If G^0 is an obstruction, then so is G.*

Proof. By Proposition 2, G^0 is isomorphic to one of $\mathbf{Z}/2 \times \mathbf{Z}/2$, D_4, D_n, or $2.S_4$. In each case, this degree-4 representation of G^0 decomposes as a direct sum of two copies of the same degree-2 representation. The generator σ of $\mathrm{Gal}(\mathbf{F}_{\ell^2}/\mathbf{F}_\ell)$ acts on these 2-dimensional subspaces by multiplication by I and $-I$, respectively, and $\mathrm{Gal}(\mathbf{F}_{\ell^2}/\mathbf{F}_\ell)$ commutes with G_1^0. Therefore, if $g \in G_1^0$ has eigenvalues $\{\lambda_1, \lambda_1, \lambda_2, \lambda_2\}$, then g^σ has eigenvalues $\{\pm\lambda_1, \pm\lambda_2\}$. The homomorphism $\psi : G_1 \longrightarrow G_2/G_3 = \{\pm I\}$ defined by $\ker\psi = \ker\psi^0 \times \langle\sigma\rangle$ shows G is an obstruction. □

Lemma 7. *If G^0 is not an obstruction and G is an obstruction, then G is a dihedral group.*

Proof. In this case, either the semisimplification of G_1^0 or G_2^0 contains the trivial representation. For the latter case, the proof of Proposition 1 applies here and implies G is a dihedral group. In the former case, the action of $\mathrm{Gal}(\mathbf{F}_{\ell^2}/\mathbf{F}_\ell)$ preserves the trivial representation, so that G cannot be an obstruction. □

5.4. **Case 3: $G_1 \subset \mathrm{GL}_2(\mathbf{F}_\ell).2$.** The maximal subgroup $\mathrm{GL}_2(\mathbf{F}_\ell).2$ of $\mathrm{Sp}_4(\mathbf{F}_\ell)$ fits into the split exact sequence

$$1 \longrightarrow \mathrm{GL}_2(\mathbf{F}_\ell) \longrightarrow \mathrm{GL}_2(\mathbf{F}_\ell).2 \xrightarrow{\pi} S_2 \longrightarrow 1,$$

where $S_2 \longrightarrow \mathrm{Out}(\mathrm{GL}_2(\mathbf{F}_\ell))$ via $g \mapsto g^*$ (inverse-transpose). The kernel $\mathrm{GL}_2(\mathbf{F}_\ell)$ of π embeds in $\mathrm{Sp}_4(\mathbf{F}_\ell)$ in 2×2 block-diagonal form as $\left(\begin{smallmatrix} g & 0 \\ 0 & g^* \end{smallmatrix}\right)$ ($g \in \mathrm{GL}_2(\mathbf{F}_\ell)$), while the non-trivial coset of $(\mathrm{GL}_2(\mathbf{F}_\ell).2)\,/\,\mathrm{GL}_2(\mathbf{F}_\ell)$ consists of matrices of the form $\left(\begin{smallmatrix} 0 & g \\ -g^* & 0 \end{smallmatrix}\right)$, with $g \in \mathrm{GL}_2(\mathbf{F}_\ell)$ [10, p. 101].

Define G^0 to be the subgroup of G corresponding to the Goursat-tuple $(\ker\pi \cap G_1, H_2, H_3, \psi|_{\ker\pi\cap G_1})$; in other words, G^0 is the subgroup of G that is embedded in 2×2 block diagonal form: $\left(\begin{smallmatrix} *\,* & & \\ *\,* & & \\ & *\,* & \\ & *\,* & \\ & & *\,* \\ & & *\,* \end{smallmatrix}\right)$. If G is an obstruction, it follows that G^0 either contains a copy of the trivial representation in its semisimplification, or is itself an obstruction. In the following lemma we show that G^0 must be an obstruction.

Proposition 3. *Let $G \subset \mathrm{Sp}_4(\mathbf{F}_\ell) \times \mathrm{SL}_2(\mathbf{F}_\ell)$ where $G_1 \subset \mathrm{GL}_2(\mathbf{F}_\ell).2$, $G_2 \subset \mathrm{SL}_2(\mathbf{F}_\ell)$, and let G^0 be as above. Then the semisimplification of G contains the trivial representation if and only if the semisimplification of G^0 does.*

Proof. If the semisimplification of G contains the trivial representation, then so does any subgroup. Conversely, if G^0 contains the trivial representation, then one of the semisimplifications of $\ker\pi \cap G_1$ or H_2 contains the trivial representation. Let $\#\ker\pi\cap G_1 = n$, and write

$$G_1 = \left\{ \begin{pmatrix} A_i & 0 \\ 0 & A_i^* \end{pmatrix}, \begin{pmatrix} 0 & B_i \\ -B_i^* & 0 \end{pmatrix} \right\},$$

where $i = 1, \ldots, n$ and $A_i, B_i \in \mathrm{GL}_2(\mathbf{F}_\ell)$.

If $\ker\pi \cap G_1$ contains the trivial representation then each A_i has 1 as an eigenvalue (or each A_i^* does, and the proof in this case is exactly the same). Therefore $\left(\begin{smallmatrix} & B_i \\ -B_i^* & \end{smallmatrix}\right)^2$ has 1 as an eigenvalue for each i, hence so does $-B_iB_i^*$. If $f_i(x)$ and $g_i(x)$ are the characteristic polynomials of $-B_iB_i^*$ and $\left(\begin{smallmatrix} 0 & B_i \\ -B_i^* & 0 \end{smallmatrix}\right)$, respectively, then $g_i(x) = f_i(x^2)$. By assumption $f_i(1) = 0$ for all i, hence $g_i(1) = 0$ for all i. This shows every $g \in G_1$ has 1 as an eigenvalue; apply Lemma 4(a) to get that G contains the trivial representation.

Next suppose the trivial representation occurs in the semisimplification of H_2. If $G_2 = H_2$ then we are done, so we may assume $[G_2 : H_2] = 2$ so that $G_2/H_2 \simeq \{\pm I\}$. In order for G to satisfy the fixed-point condition, every $\begin{pmatrix} 0 & B_i \\ -B_i^* & 0 \end{pmatrix} \in G_1$ has 1 as an eigenvalue. By the preceding argument, this forces all of G_1 to have 1 as an eigenvalue, contradicting Lemma 4(a) and finishing the proof. □

Now suppose G^0 is an obstruction. Proposition 2 can be adapted to this case to show G^0 is isomorphic to one of $\mathbf{Z}/2 \times \mathbf{Z}/2$, D_4, $2.S_4$, or D_n. It remains to solve the extension problem

$$1 \longrightarrow G^0 \longrightarrow G \longrightarrow S_2 \longrightarrow 1$$

for obstructions G^0 and G.

Proposition 4. *With all notation as in Proposition 3, if G and G^0 are obstructions then $G_2/G_3 = \{\pm I\}$ and*

(i) $G^0 \cap G_1 \simeq \mathbf{Z}/2 \times \mathbf{Z}/2$, *and* $G_1 \simeq D_4$, *or*
(ii) $G^0 \cap G_1 \simeq D_4$, *and* $G_1 \simeq D_4.2$, *or*
(iii) $G^0 \cap G_1 \simeq 2.S_4$, *and* $G_1 \simeq 2.S_4.2$, *or*
(iv) $G^0 \cap G^1 \simeq D_n$, *and* $G_1 \simeq D_n.2$.

Proof. Write $\ker \pi \cap G_1 = H^0.2$, where $H^0 \simeq \mathbf{Z}/2$, C_4, $\mathrm{SL}_2(\mathbf{F}_3)$, or C_n (a cyclic group of order n) when $\ker \pi \cap G_1 \simeq \mathbf{Z}/2 \times \mathbf{Z}/2$, D_4, $\mathrm{GL}_2(\mathbf{F}_3)$, or D_n respectively. In each case, the homomorphism ϕ of the Goursat-tuple attached to G^0 is defined by $\ker \phi = H^0$.

If we write $G_1 = G^0.2 = H^0.\langle\tau\rangle.\langle\sigma\rangle$ (so that τ and σ each generate a group of order 2), then this means $\ker \psi = \{H^0, H^0.\langle\sigma\rangle\}$.

Therefore G_2/G_3 has order 2 and one checks that the fixed-point condition holds for all G. This proves the proposition. □

5.5. **Case 4:**$G_1 \subset 2.2^4.\mathrm{O}_4^-(\mathbf{F}_2)$**.** Here we suppose G_1 normalizes the extra-special 2-group 2.2^4, which is isomorphic to the central product $D_4 \circ Q_8$ [10, p. 153-154] (where D_4 and Q_8 are the dihedral and quaternion groups of order 8, respectively). Each group has an irreducible degree-2 representation:

$$D_4 = \left\langle x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle$$
$$Q_8 = \left\langle x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, y = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \right\rangle,$$

where $a, b \in \mathbf{F}_\ell$ satisfy $a^2 + b^2 = -1$. The centers of D_4 and Q_8 each have order 2, so the central product $D_4 \circ Q_8$ is with respect to the non-trivial homomorphism $Z(D_4) \longrightarrow Z(Q_8)$.

There is an embedding $2.2^4 \hookrightarrow \mathrm{Sp}_4(\mathbf{F}_\ell)$ given by the image of the tensor product representation of the two irreducible representations above [10, p. 151]. Of the 32 elements of 2.2^4, there are 14 without 1 as an eigenvalue in this representation. The subgroup generated by these 14 elements is all of 2.2^4.

If $\ell \equiv \pm 1(8)$, then the normalizer of 2.2^4 in $\mathrm{Sp}_4(\mathbf{F}_\ell)$ is $\mathrm{O}_4^-(\mathbf{F}_2)$; otherwise it is $\Omega_4^-(\mathbf{F}_2)$, where $\Omega_4^-(\mathbf{F}_2) \simeq A_5$ is the unique index-2 subgroup of $\mathrm{O}_4^-(\mathbf{F}_2) \simeq S_5$ [10, § 2.5], [10, p. 44]. The following lemmas show that neither $2.2^4.\mathrm{O}_4^-(\mathbf{F}_2)$ nor $2.2^4.\Omega_4^-(\mathbf{F}_2)$ are obstructions.

Lemma 8. *Let $G \subset \mathrm{Sp}_4(\mathbf{F}_\ell) \times \mathrm{SL}_2(\mathbf{F}_\ell)$ be given by the Goursat-tuple (G_1, G_2, G_3, ψ), with $G_1 = 2.2^4.\Omega_4^-(\mathbf{F}_2)$. Then G is not an obstruction.*

Proof. Suppose G were an obstruction. Since the subgroup of 2.2^4 generated by the elements without 1 as an eigenvalue is all of 2.2^4, $\ker \psi$ contains 2.2^4. Since $(2.2^4.\Omega_4^-(\mathbf{F}_2))/2.2^4 \simeq A_5$ is simple, either $\ker \psi = 2.2^4$ or $\ker \psi = 2.2^4.\Omega_4^-(\mathbf{F}_2)$. The former case is impossible since A_5 is not a subgroup of $\mathrm{SL}_2(\mathbf{F}_\ell)$. In the latter case Lemma 1(b) implies $G_2 = G_3$ consists entirely of elements having 1 as an eigenvalue, contradicting the assumption that G is an obstruction. □

Lemma 9. *Let $G \subset \mathrm{Sp}_4(\mathbf{F}_\ell) \times \mathrm{SL}_2(\mathbf{F}_\ell)$ be given by the Goursat-tuple (G_1, G_2, G_3, ψ), with $G_1 = 2.2^4.\mathrm{O}_4^-(\mathbf{F}_2)$ (so $\ell \equiv \pm 1(8)$). Then G is not an obstruction.*

Proof. Fix a square root α of 2. By the proof of Lemma 7, if G were an obstruction, then $\ker \psi = 2.2^4.\Omega_4^-(\mathbf{F}_2)$. The matrix

$$g = \alpha^{-1} \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

is an element of $2.2^4.\mathrm{O}_4^-(\mathbf{F}_2) - 2.2^4.\Omega_4^-(\mathbf{F}_2)$ [10, p. 155] which does not have 1 as an eigenvalue. It follows that $\ker\psi = G_1$ and $G_2 = G_3$ consists entirely of elements having 1 as an eigenvalue, contradicting the assumption that G is an obstruction. □

5.6. **Case 5: The Exotic Subgroups A_6, S_6, and $\mathrm{PSL}_2(\mathbf{F}_\ell)$.** Recall that the *exotic* subgroups (type $\mathcal{S}$ in [10]) of a finite group of Lie type are maximal subgroups which do not fall into the any of the classes $\mathcal{C}_i$, $i = 1,\ldots,8$. According to [9, p. 209], the exotic subgroups of $\mathrm{PSp}_4(\mathbf{F}_\ell)$ when $\ell \geq 7$ are A_6 ($\ell \equiv \pm 5$ (12)), S_6 ($\ell \equiv \pm 1$ (12)), and $\mathrm{PSL}_2(\mathbf{F}_\ell)$ (via the third symmetric power representation). These $\mathbf{F}_\ell$-representations of A_6 and S_6 are the reductions mod ℓ of the ordinary degree-4 representations (since $\ell \geq 7$). The preimages under the map $\pi : \mathrm{Sp}_4(\mathbf{F}_\ell) \longrightarrow \mathrm{PSp}_4(\mathbf{F}_\ell)$ of A_6 and S_6 are non-trivial double covers denoted by $2.A_6$ and $2.S_6$ respectively.

The Schur Multiplier of $\mathrm{PSL}_2(\mathbf{F}_\ell)$ has order 2 and $\mathrm{PSL}_2(\mathbf{F}_\ell)$ is perfect, so by [7, thm. 2.1.19] there are only two inequivalent, degree-2 central extensions of $\mathrm{PSL}_2(\mathbf{F}_\ell)$: the trivial extension and $\mathrm{SL}_2(\mathbf{F}_\ell)$.

Next we recall a theorem of Schur on the double covers of S_n and then finish this section by analyzing the subgroups $2.A_6$, $2.S_6$, and $\mathrm{SL}_2(\mathbf{F}_\ell)$ of $\mathrm{Sp}_4(\mathbf{F}_\ell)$.

Theorem 2 (Schur)**.** [7, p. 103] *Given $n \geq 4$, define the groups S_n^* and S_n^{**} as follows:*

$$S_n^* = \langle g_1,\ldots,g_{n-1}, z : g_i^2 = (g_i g_{i+1})^3 = (g_k g_l)^2 = z, z^2 = [z, g_i] = 1\rangle$$
$$S_n^{**} = \langle g_1,\ldots,g_{n-1}, z : g_i^2 = (g_i g_{i+1})^3 = 1, (g_k g_l)^2 = z, z^2 = [z, g_i] = 1\rangle$$
$$(1 \leq i \leq n-1,\ 1 \leq j \leq n-2,\ k \leq l-2)$$

If $n \geq 4$ and $n \neq 6$, then there exist exactly two non-isomorphic covering groups of S_n, namely S_n^* and S_n^{**}, defined above; S_6^* is the only (up to isomorphism) covering group of S_6.

We omit the proof of the following lemma.

Lemma 10. *Let H be a subgroup of S_6 such that $\#H \leq 60$, and H contains $\geq \#H/2$ transpositions. Then H contains 1, 2, or 3 transpositions and is isomorphic to S_2, $S_2 \times S_2$, or S_3 respectively.*

Proposition 5. *Suppose $G \subset \mathrm{Sp}_4(\mathbf{F}_\ell) \times \mathrm{SL}_2(\mathbf{F}_\ell)$ is an obstruction given by the Goursat-tuple (G_1, G_2, G_3, ψ) with $G_1 \subset 2.S_6 \subset \mathrm{Sp}_4(\mathbf{F}_\ell)$. Then G is isomorphic to one of $S_2 \times S_2$, $2.(S_2 \times S_2)$, S_3, or $2.S_3$.*

Proof. First, if $G_1 \subset 2.A_6$ then the only element of the degree-4 representation of G_1 having 1 as an eigenvalue is the identity (Appendix A). By Lemma 4(b), G cannot be an obstruction.

According to Appendix A, the only non-trivial elements of the degree-4 representation of $2.S_6$ having 1 as an eigenvalue belong to the conjugacy class $2B_0$. Let $\pi : 2.S_6 \longrightarrow S_6$ be the natural projection. The conjugacy class $2B_0$ of $2.S_6$ is induced from the conjugacy class $2B$ of S_6 which contains the transpositions. If z generates the center of $2.S_6$, and $g_i \in 2B_0$, then zg_i has 1 as an eigenvalue also since the characteristic polynomials of z and g_i are $(x+1)^4$ and $(x-1)^2(x+1)^2$ respectively, whence $g_i z \in 2B_0$. It follows that $\#2B_0 = 30$. Moreover, the relations of Theorem 2 tell us that the product of any two disjoint transpositions in S_6 lifts to an element of order 4 in $2.S_6$ and therefore does not have eigenvalue 1. By inequality (1), $\#G_1$ is at most 60:

$$\frac{\#G_1}{2} \geq \frac{\#G_1}{[G_2 : G_3]} \geq \underbrace{\#G_1 - 31}_{\#P} + 1 \geq \#G_1 - 30.$$

Set $\overline{G_1} := \pi(G_1)$ so that $\#G_1 = \#\overline{G_1}$, or $\#G_1 = 2\#\overline{G_1}$. In either case $\overline{G_1}$ defines a subgroup of S_6 with the property that $\overline{G_1}$ contains $\geq \#\overline{G_1}/2$ transpositions and $\#\overline{G_1} \leq 60$. Lemma 10 implies $\overline{G_1}$ is isomorphic to one of S_2, $S_2 \times S_2$, or S_3.

Suppose G_1 does not contain the center $Z = \langle z\rangle$ of $2.S_6$, so that $G_1 \simeq \overline{G_1}$. If $G_1 \simeq S_2$, then by Lemma 4(a) G cannot be an obstruction. If $G_1 \simeq S_2 \times S_2$, then we get an obstruction G with Goursat-tuple (G_1, G_2, G_3, ψ) by defining $\ker\psi$ to be generated by the product of the two disjoint transpositions in G_1. Lastly, if $G_1 \simeq S_3$, then we get another obstruction G with Goursat-tuple (G_1, G_2, G_3, ψ) by defining $\ker\psi \simeq A_3$.

It remains to consider the cases where $Z \subset G_1$, so that $\#G_1 = 2\#\overline{G_1}$. If $\overline{G_1} \simeq S_2$, then $G_1 \simeq S_2 \times Z$, and the homomorphism $\psi : G_1 \longrightarrow \{\pm I\}$ with $\ker\psi = Z$ defines an obstruction. Next, if $\overline{G_1} \simeq S_2 \times S_2$, then G_1 has order 8 and contains an element of order 4, namely the preimage of the product of the disjoint transpositions. We obtain an obstruction G of order 8 with Goursat-tuple (G_1, G_2, G_3, ψ) by defining $\ker\psi$ to be cyclic of order 4.

Finally, suppose that $\overline{G_1} \simeq S_3$. The composite homomorphism

$$G_1 \xrightarrow{\pi} G_1/Z \simeq S_3 \xrightarrow{\epsilon} \{\pm I\}$$

defines an obstruction $G \simeq 2.S_3$, which finishes the proof of the proposition. □

The group $\mathrm{PSL}_2(\mathbf{F}_\ell)$ does not have a degree-4 $\mathbf{F}_\ell$-representation when $\ell \geq 7$, which means the central extension $2.\mathrm{PSL}_2(\mathbf{F}_\ell)$ is isomorphic to $\mathrm{SL}_2(\mathbf{F}_\ell)$. The following proposition shows that no obstruction $G \subset \mathrm{Sp}_4(\mathbf{F}_\ell) \times \mathrm{SL}_2(\mathbf{F}_\ell)$ can have $G_1 \subset \mathrm{SL}_2(\mathbf{F}_\ell)$.

Proposition 6. *Let $G \subset \mathrm{Sp}_4(\mathbf{F}_\ell) \times \mathrm{SL}_2(\mathbf{F}_\ell)$ be given by the Goursat-tuple (G_1, G_2, G_3, ψ), with $G_1 \subset \mathrm{Sym}^3(\mathrm{SL}_2(\mathbf{F}_\ell))$. Then G is not an obstruction.*

Proof. The exotic subgroup $\mathrm{SL}_2(\mathbf{F}_\ell)$ of $\mathrm{Sp}_4(\mathbf{F}_\ell)$ is the image of third symmetric-power representation Sym^3 [9, p. 70]. If $g \in \mathrm{SL}_2(\mathbf{F}_\ell)$ has eigenvalues $\lambda^{\pm 1}$ (acting on it's natural module), then $\mathrm{Sym}^3(g)$ has eigenvalues $\lambda^{\pm 3}, \lambda^{\pm 1}$. Therefore, an element $\mathrm{Sym}^3(g)$ of $\mathrm{Sym}^3(\mathrm{SL}_2(\mathbf{F}_\ell))$ has 1 as an eigenvalue if and only if g the eigenvalues of g are third roots of unity.

Let $\mathcal{G}_1 \subset \mathrm{SL}_2(\mathbf{F}_\ell)$ be the subgroup satisfying $\mathrm{Sym}^3(\mathcal{G}_1) = G_1$, and let N be the number of elements of $\mathcal{G}_1$ of order 3.

Suppose G were an obstruction, so that $N \geq \#\mathcal{G}_1/2$. If $\mathcal{G}_1$ is Cartan, then $N|3$ and $\#\mathcal{G}_1|6$. When $\#\mathcal{G}_1 = 6$, a generator of $\mathcal{G}_1$ does not have 1 as an eigenvalue, hence is in $\ker\psi$ (so ψ is trivial), contradicting the assumption that G be an obstruction. On the other hand, if $\#\mathcal{G}_1 = 3$, then every element of G_1 has 1 as an eigenvalue, which contradicts Lemma 4(a).

If $\mathcal{G}_1$ is a Borel subgroup of $\mathrm{SL}_2(\mathbf{F}_\ell)$, then its eigenvalues are the same as those of its Cartan subgroup. There are at most 3ℓ elements of $\mathcal{G}$ having eigenvalues which are 3^{rd} roots of unity, which gives us $\#\mathcal{G}_1|6\ell$. Just as in the Cartan case, we do not get an obstruction.

If $\mathcal{G}_1$ normalizes a Cartan subgroup of $\mathrm{SL}_2(\mathbf{F}_\ell)$, then any element of order 3 necessarily lies in the Cartan subgroup. Therefore $\mathcal{G}_1$ has at most three elements of order 3, forcing $\#\mathcal{G}_1|6$. The normalizer of a Cartan subgroup of $\mathrm{SL}_2(\mathbf{F}_\ell)$ contains the element $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, which has order 4 and does not have 1 as an eigenvalue. This contradicts $\#\mathcal{G}_1|6$, hence $\mathcal{G}_1$ cannot simultaneously normalize a Cartan subgroup of $\mathrm{SL}_2(\mathbf{F}_\ell)$ and satisfy the fixed-point condition.

Finally suppose $\mathcal{G}_1$ is an exotic subgroup of $\mathrm{SL}_2(\mathbf{F}_\ell)$. The groups $2.A_4$ and $2.S_4$ each have 8 elements of order 3, while $2.A_5$ has 20 elements of order 3. None of these groups satisfy $N \geq \#\mathcal{G}_1/2$, hence cannot be obstructions. This finishes the proof of the proposition. □

6. Subgroups of $\mathrm{SL}_2(\mathbf{F}_\ell) \wr S_3$

Recall from Section 1.3 that the maximal subgroup $\mathrm{SL}_2(\mathbf{F}_\ell) \wr S_3$ of $\mathrm{Sp}_6(\mathbf{F}_\ell)$ stabilizes the decomposition of a six-dimensional symplectic space into a sum of hyperbolic planes. The subgroup $\mathrm{SL}_2(\mathbf{F}_\ell) \times \mathrm{SL}_2(\mathbf{F}_\ell) \times \mathrm{SL}_2(\mathbf{F}_\ell)$ (henceforth denoted $\mathrm{SL}_2(\mathbf{F}_\ell)^3$) embeds in $\mathrm{Sp}_6(\mathbf{F}_\ell)$ as block-diagonal matrices. The natural S_3-action on the SL_2-factors defines a splitting of the exact sequence

$$1 \longrightarrow \mathrm{SL}_2(\mathbf{F}_\ell)^3 \longrightarrow \mathrm{SL}_2(\mathbf{F}_\ell) \wr S_3 \xrightarrow{\pi} S_3 \longrightarrow 1.$$

For the rest of this section let $G \subset \mathrm{SL}_2(\mathbf{F}_\ell) \wr S_3$ be an obstruction and set $G^0 := G \cap \ker\pi$. Since G^0 is a subgroup of a triple direct product it can be described by two applications of Goursat's lemma: write $G^0 = (A_1 \bullet A_2) \bullet A_3$, relative to the Goursat-tuples

$$(A_1 \bullet A_2, A_3, B_3, \psi) \text{ and } (A_1, A_2, B_2, \phi).$$

Since G satisfies the fixed-point condition, so does G^0, whence G^0 is either an obstruction itself, or contains a copy of the trivial representation in its semisimplification. The following lemma applies to the latter case.

Lemma 11. *Let $G^0 \subset \mathrm{SL}_2(\mathbf{F}_\ell)^3$ have a copy of the trivial representation in its semisimplification. Then one of the A_i is either trivial, or has order ℓ.*

Proof. By assumption there exists a basis for $\mathbf{F}_\ell^6$ such that one of the A_i is of the form $\left(\begin{smallmatrix}1 & *\\ 0 & 1\end{smallmatrix}\right)$, since we can change basis using block-diagonal matrices. $\square$

We divide this section into four parts, based on the image of G/G^0 in S_3, and begin with the case $G = G^0$.

6.1. **Case 1:** $G \subset \mathrm{SL}_2(\mathbf{F}_\ell)^3$**.** We assume $G = G^0$ and employ all the notation above, so that $G = (A_1 \bullet A_2) \bullet A_3$.

Lemma 12. *If* $G = G^0 \subset \mathrm{SL}_2(\mathbf{F}_\ell)^3$ *is an obstruction, then* $G \simeq \mathbf{Z}/2 \times \mathbf{Z}/2$*.*

Proof. Observe that G is necessarily a Goursat-subgroup of $\mathrm{SL}_2(\mathbf{F}_\ell)^3$: otherwise each $A_i \subset \mathrm{SL}_2(\mathbf{F}_\ell)$ would consist entirely of elements with eigenvalues 1; by [12, p. I-2, ex. 1] such a group is not an obstruction. Therefore, write $G = (A_1 \bullet A_2) \bullet A_3$. Set $\#A_1 = n$, $\#B_2 = m$, and $[A_3 : B_3] = k$. By inequality (1) we have

$$\frac{nm}{k} \geq nm - \#\ker\phi - m + 1 + 1,$$

and since $\#\ker\phi \leq n$, Lemma 3 assures us that $n = m = k = 2$ and $G \simeq \mathbf{Z}/2 \times \mathbf{Z}/2$, as desired. $\square$

Remark. The reason this is the "only" obstruction G is because we are tacitly assuming the remark following the proof of Lemma 1. More generally, we could take each A_i to be the Borel subgroup $\mathbf{Z}/2 \ltimes \mathbf{Z}/\ell$ and define ψ in the same way.

6.2. **Case 2:** $G \subset \mathrm{SL}_2(\mathbf{F}_\ell) \wr \mathbf{Z}/3$**.** Here we assume $G/G^0 \simeq \mathbf{Z}/3$, and since G is an obstruction, we know G^0 satisfies the fixed-point assumption. Therefore G^0 is either an obstruction or, by Lemma 11, at least one of the A_i is either trivial or has order ℓ.

Lemma 13. *Suppose* G^0 *is an obstruction, so* $G^0 \simeq \mathbf{Z}/2 \times \mathbf{Z}/2$*. Then* $G \simeq A_4$*.*

Proof. We have an exact sequence

$$0 \longrightarrow \mathbf{Z}/2 \times \mathbf{Z}/2 \longrightarrow G \longrightarrow \mathbf{Z}/3 \longrightarrow 0$$

with G non-abelian (it contains elements of the form $\left(\begin{smallmatrix}0 & a & 0\\ 0 & 0 & b\\ c & 0 & 0\end{smallmatrix}\right)$, $a, b, c \in \mathrm{SL}_2(\mathbf{F}_\ell)$). Hence $G \simeq A_4$ and embeds in $\mathrm{Sp}_6(\mathbf{F}_\ell)$ via two copies of the standard irreducible 3-dimensional representation of A_4. $\square$

If G^0 is not an obstruction, then by Lemma 11 we may assume without loss of generality that there exists a basis so that $A_3 = \langle\left(\begin{smallmatrix}1 & *\\ 0 & 1\end{smallmatrix}\right)\rangle$. Let $\#G^0 = N$ and write

$$G = \left\{ \begin{pmatrix} a_{1_j} & 0 & 0 \\ 0 & a_{2_j} & 0 \\ 0 & 0 & a_{3_j} \end{pmatrix}, \begin{pmatrix} 0 & b_{1_j} & 0 \\ 0 & 0 & b_{2_j} \\ b_{3_j} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & c_{1_j} \\ c_{2_j} & 0 & 0 \\ 0 & c_{3_j} & 0 \end{pmatrix} \right\},$$

where $1 \leq j \leq N$ and a_{3_j} has order dividing ℓ for all j. We will now describe the G which are *not* obstructions.

Proposition 7. *Suppose* G *satisfies the fixed-point condition and that* G^0 *is not an obstruction. Then* G *is not an obstruction if and only if* G *is of the form*

$$G = \left\{ \begin{pmatrix} 1 & * & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & * & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \alpha & * & 0 & 0 \\ 0 & 0 & 0 & \alpha^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta & * \\ 0 & 0 & 0 & 0 & 0 & \beta^{-1} \\ (\alpha\beta)^{-1} & * & 0 & 0 & 0 & 0 \\ 0 & \alpha\beta & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & \alpha\beta & * \\ 0 & 0 & 0 & 0 & 0 & (\beta\alpha)^{-1} \\ \alpha^{-1} & * & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta^{-1} & * & 0 & 0 \\ 0 & 0 & 0 & \beta & 0 & 0 \end{pmatrix} \right\}, \ \alpha, \beta \in \mathbf{F}_\ell^\times.$$

Proof. Sufficiency is immediate since G fixes $v = (\beta, 0, \alpha^{-1}\beta, 0, \alpha^{-1}, 0)$. Conversely, if G has a copy of the trivial representation in its semisimplification, then so does G^0. Without loss of generality we may assume A_3 has order dividing ℓ, by Lemma 11. Since G^0 satisfies the fixed-point condition, its eigenvectors are of the form $v = (x, 0, y, 0, z, 0)$, where $z \in \mathbf{F}_\ell^\times$, and $x, y \in \mathbf{F}_\ell$ are non-zero if and only if A_2 and A_3 have order dividing ℓ, respectively. However, if either x or y is zero, then no element of $G \setminus G^0$ can fix v. We can therefore assume that each A_i has order dividing ℓ, which forces G to be of the form above. $\square$

It follows that any $G \subset \mathrm{SL}_2(\mathbf{F}_\ell) \wr \mathbf{Z}/3$ which satisfies the fixed-point condition and is not of this form is an obstruction. For example, the group $G \simeq \mathrm{SL}_2(\mathbf{F}_\ell) \rtimes \mathbf{Z}/3$ defined by

$$G \cap \ker \pi = \left\{ \left(\begin{smallmatrix} A & 0 & 0 \\ 0 & A^{-1} & 0 \\ 0 & 0 & I \end{smallmatrix} \right) \right\},$$

where $A \in \mathrm{SL}_2(\mathbf{F}_\ell)$, together with the standard $\mathbf{Z}/3$-action on the diagonal factors, is not of the form in Proposition 7, yet it satisfies the fixed-point condition.

6.3. **Case 3: $G \subset \mathrm{SL}_2(\mathbf{F}_\ell) \wr S_3$.** Next, suppose $G/G^0 \simeq S_3$, and that G is an obstruction. It follows that G^0 is either an obstruction, or contains a copy of the trivial representation in its semisimplification.

Lemma 14. *If G^0 is an obstruction, then G is an obstruction if and only if $G \simeq S_4$.*

Proof. By Lemma 12 $G^0 \simeq \mathbf{Z}/2 \times \mathbf{Z}/2$, hence G is a non-abelian group of order 24 with $G/G^0 \simeq S_3$. The explicit embedding of G in $\mathrm{Sp}_6(\mathbf{F}_\ell)$ makes it straightforward to show that G satisfies the fixed-point condition (and is therefore an obstruction) if and only if $G \simeq S_4$. □

Next we consider the case where the semisimplification of G^0 contains the trivial representation. In the following proposition we give necessary and sufficient conditions for G *not* to be an obstruction. Hence, any G which is not of that form and satisfies the fixed-point condition is necessarily an obstruction; an example of such G follows the proof of the proposition.

Lemma 15. *Suppose G^0 contains a copy of the trivial representation in its semisimplification, so is not an obstruction. Then G is not an obstruction if and only if G is of the form*

$$G = \left\{ \begin{pmatrix} a_{1_j} & 0 & 0 \\ 0 & a_{2_j} & 0 \\ 0 & 0 & a_{3_j} \end{pmatrix}, \begin{pmatrix} 0 & b_{1_j} & 0 \\ 0 & 0 & b_{2_j} \\ b_{3_j} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & c_{1_j} \\ c_{2_j} & 0 & 0 \\ 0 & c_{3_j} & 0 \end{pmatrix}, \begin{pmatrix} e_{1_j} & 0 & 0 \\ 0 & 0 & e_{2_j} \\ 0 & e_{3_j} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & f_{1_j} \\ 0 & f_{2_j} & 0 \\ f_{3_j} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & g_{1_j} & 0 \\ g_{2_j} & 0 & 0 \\ 0 & 0 & g_{3_j} \end{pmatrix} \right\},$$

where $a_{i_j} = e_{1_j} = f_{2_j} = g_{3_j} = \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix} \right)$ for all i, j, and

$$\prod_i b_{i_j} = \prod_i c_{i_j} = \prod_i e_{i_j} = \prod_i f_{i_j} = \prod_i g_{i_j} = \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix} \right).$$

Proof. The proof is nearly identical to that of Lemma 7, so we omit it. □

An example of an obstruction G where G^0 is not a obstruction is the following. Let J generate a Sylow-ℓ subgroup of $\mathrm{SL}_2(\mathbf{F}_\ell)$ and define:

$$G = \left\{ \left(\begin{smallmatrix} J & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & J \end{smallmatrix} \right), \left(\begin{smallmatrix} 0 & J & 0 \\ 0 & 0 & J \\ J & 0 & 0 \end{smallmatrix} \right), \left(\begin{smallmatrix} 0 & 0 & J \\ J & 0 & 0 \\ 0 & J & 0 \end{smallmatrix} \right), \left(\begin{smallmatrix} -J & 0 & 0 \\ 0 & 0 & -J \\ 0 & -J & 0 \end{smallmatrix} \right), \left(\begin{smallmatrix} 0 & 0 & -J \\ 0 & -J & 0 \\ -J & 0 & 0 \end{smallmatrix} \right), \left(\begin{smallmatrix} 0 & -J & 0 \\ -J & 0 & 0 \\ 0 & 0 & -J \end{smallmatrix} \right) \right\}.$$

6.4. **Case 4: $G \subset \mathrm{SL}_2(\mathbf{F}_\ell) \wr S_2 \times \mathrm{SL}_2(\mathbf{F}_\ell)$.** Here we assume $G/G^0 \simeq S_2$ and start with the case where G^0 is an obstruction.

Lemma 16. *Suppose $G^0 \simeq \mathbf{Z}/2 \times \mathbf{Z}/2$ and G is an obstruction. Then $G \simeq D_4$.*

Proof. Without loss of generality we may assume the S_2-action on G^0 interchanges the first and second components. The fixed-point condition holds for G, which is a non-abelian group of order 8 with a normal subgroup isomorphic to $\mathbf{Z}/2 \times \mathbf{Z}/2$. Therefore $G \simeq D_4$ and is an obstruction. □

When G^0 is not an obstruction, let $\#G^0 = n$ and without loss of generality write $G^0 = \left\{ \begin{pmatrix} a_{1_i} & 0 & 0 \\ 0 & a_{2_i} & 0 \\ 0 & 0 & a_{3_i} \end{pmatrix} \right\}$, where a_{3_i} has order dividing ℓ for all i. The S_2-action can either interchange the first two components, or it can interchange the third component with the either of the first two. In the former case, G defines a subgroup of $\mathrm{Sp}_4(\mathbf{F}_\ell) \times \mathrm{SL}_2(\mathbf{F}_\ell)$, and we have the following lemma:

Lemma 17. *Let G be given by the Goursat-tuple (G_1, G_2, G_3, ψ), with $G_1 \subset \mathrm{SL}_2(\mathbf{F}_\ell) \wr S_2 \subset \mathrm{Sp}_4(\mathbf{F}_\ell)$, and $G_3 \lhd G_2 \subset \mathrm{SL}_2(\mathbf{F}_\ell)$. Suppose further that $\#G_2^0 = \ell$. Then G is an obstruction if and only if $G \simeq D_n$, with $n|(\ell^2 - 1)$.*

Proof. Choose a basis so that G has the form

$$G = \left\{ \begin{pmatrix} a_{1_i} & 0 & 0 \\ 0 & a_{2_i} & 0 \\ 0 & 0 & a_{3_i} \end{pmatrix}, \begin{pmatrix} 0 & b_{1_i} & 0 \\ b_{2_i} & 0 & 0 \\ 0 & 0 & b_{3_i} \end{pmatrix} \right\},$$

where $a_{3_i} = \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ for each i. If G is an obstruction, then $G_2 = \left\{ \left(\begin{smallmatrix} -1 & * \\ 0 & -1 \end{smallmatrix}\right) \right\}$, so that $G_2/G_3 \simeq \{\pm I\}$.

Since none of the b_{3_i} have 1 as an eigenvalue, it must be the case that the elements $\begin{pmatrix} 0 & b_{1_i} \\ b_{2_i} & 0 \end{pmatrix}$ of G_1 each have 1 as an eigenvalue. The characteristic polynomial of $\begin{pmatrix} 0 & b_{1_i} \\ b_{2_i} & 0 \end{pmatrix}$ is

$$f(x) = x^4 - \alpha x^2 + 1,$$

where α is a polynomial in the entries of b_{1_i} and b_{2_i}. We have assumed and $f(1) = 0$, hence $\alpha = 2$. Therefore, the *minimal* polynomial is $x^2 - 1$, *i.e.* $b_{1_i} = b_{2_i}^{-1}$ for all i.

For ease of notation, rewrite G in the following form:

$$G = \left\{ \begin{pmatrix} b_1 b_i^{-1} & 0 \\ 0 & b_1^{-1} b_i \end{pmatrix}, \begin{pmatrix} 0 & b_i \\ b_i^{-1} & 0 \end{pmatrix} \right\}, \ i = 1, \ldots, n.$$

Let G_1 be the index-2 subgroup of G_1 consisting of all the $\begin{pmatrix} b_1 b_i^{-1} & 0 \\ 0 & b_1^{-1} b_i \end{pmatrix}$. Then G_1^0 is a subgroup of $\mathrm{SL}_2(\mathbf{F}_\ell) \times \mathrm{SL}_2(\mathbf{F}_\ell)$, hence is given by a Goursat-tuple (H_1, H_2, H_3, ψ). The explicit description of G_1^0 shows that H_3 is trivial and $\# H_2 = \# H_1$. The map $\psi : H_1 \longrightarrow H_2/H_3$ defined by $\psi(b_1 b_j^{-1}) = b_1^{-1} b_j$ is a homomorphism if and only if H_1 is abelian. Therefore, G_1 is isomorphic to a dihedral group D_n with $n | (\ell^2 - 1)$, which finishes the proof. □

It remains to analyze the case where the S_2-action interchanges the third component with either of the first two (recall a_{3_i} is assumed to have order dividing ℓ for all i). Without loss of generality we assume that the second and third components are permuted.

Lemma 18. *Let $G \subset \mathrm{SL}_2(\mathbf{F}_\ell) \times (\mathrm{SL}_2(\mathbf{F}_\ell) \wr S_2)$ and $G^0 \subset G$ be as above so that a_{3_i} has order dividing ℓ for all i. Then G cannot be an obstruction.*

Proof. Write $G = \left\{ \begin{pmatrix} a_{1_i} & 0 & 0 \\ 0 & a_{2_i} & 0 \\ 0 & 0 & a_{3_i} \end{pmatrix}, \begin{pmatrix} b_{1_i} & 0 & 0 \\ 0 & 0 & b_{2_i} \\ 0 & b_{3_i} & 0 \end{pmatrix} \right\}$, so that the product $b_{3_i} b_{2_i}$ has order dividing ℓ for all i and choose a basis so that $a_{3_i} = \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ for all i. It follows that $b_{2_i} = \left(\begin{smallmatrix} x & * \\ -y & * \end{smallmatrix}\right)$ and $B_{3_i} = \left(\begin{smallmatrix} * & * \\ y & x \end{smallmatrix}\right)$, for fixed $x, y \in \mathbf{F}_\ell$, not both zero.

Pick any $A = \begin{pmatrix} a_{1_{j_0}} & 0 & 0 \\ 0 & a_{2_{j_0}} & 0 \\ 0 & 0 & a_{3_{j_0}} \end{pmatrix} \in G^0$ with $a_{2_{j_0}} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ non-trivial. Left multiplication by A and A^{-1} lead to the equations:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \left(\begin{smallmatrix} x & * \\ -y & * \end{smallmatrix}\right) = \left(\begin{smallmatrix} x & * \\ -y & * \end{smallmatrix}\right)$$

$$\begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} \left(\begin{smallmatrix} x & * \\ -y & * \end{smallmatrix}\right) = \left(\begin{smallmatrix} x & * \\ -y & * \end{smallmatrix}\right),$$

from which it follows that $\operatorname{tr} a_{2_{j_0}} = 2$. Thus a_{2_j} has order dividing ℓ for all j, hence so does $b_{2_j} b_{3_j}$. It follows that there exists $z \in \mathbf{F}_\ell^\times$ such that

$$b_{2_j} = \left(\begin{smallmatrix} z & * \\ 0 & z^{-1} \end{smallmatrix}\right) \text{ and } b_{3_j} = \left(\begin{smallmatrix} z^{-1} & * \\ 0 & z \end{smallmatrix}\right)$$

for all j. But this means G fixes the line spanned by $(0, 0, z, 0, 1, 0)$ and cannot be an obstruction. □

7. Subgroups of $\mathrm{GL}_3(\mathbf{F}_\ell).2$

Define the *semisimplification type* of a linear group G to be the tuple whose entries are the dimensions of the irreducible subspaces with respect to the semisimplification of G. Hence, any subgroup of $\mathrm{GL}_3(\mathbf{F}_\ell)$ has semisimplification type $(1,1,1)$, $(2,1)$ or (3).

If G is any subgroup of $\mathrm{GL}_3(\mathbf{F}_\ell).2$, then define $G^0 := G \cap \ker \pi \subset \mathrm{GL}_3(\mathbf{F}_\ell)$ relative to the split exact sequence

$$1 \longrightarrow \mathrm{GL}_3(\mathbf{F}_\ell) \longrightarrow \mathrm{GL}_3(\mathbf{F}_\ell).2 \xrightarrow{\pi} S_2 \longrightarrow 1 \ ,$$

so that $G^0 \hookrightarrow \mathrm{Sp}_6(\mathbf{F}_\ell)$ in block-diagonal form.

The obstructions $G \subset \mathrm{GL}_3(\mathbf{F}_\ell).2$ fall naturally into two categories: those for which G^0 is also an obstruction, and those for which it is not. In the first case, those G for which G^0 has semisimplification type $(1,1,1)$ or $(2,1)$ are listed in Lemma 3 and Proposition 2, while those with semisimplification type (3) will be described shortly. We then solve the group extension problem

$$1 \longrightarrow G^0 \longrightarrow G \longrightarrow S_2 \longrightarrow 1,$$

and also describe all obstructions G for which G^0 is *not* an obstruction.

7.1. **Irreducible $\mathrm{GL}_3(\mathbf{F}_\ell)$-Obstructions.** Let $G^0 \subset \mathrm{GL}_3(\mathbf{F}_\ell)$ have semisimplification type (3) so that G^0 acts irreducibly on $\mathbf{F}_\ell^3$; by Clifford's theorem $G^0 \cap \mathrm{SL}_3(\mathbf{F}_\ell)$ is either irreducible or has semisimplification type $(1,1,1)$; we start by assuming the former. According to [2, p. 170-177] and [9, p. 191], the irreducible proper subgroups of $\mathrm{SL}_3(\mathbf{F}_\ell)$ (when $\ell \geq 7$) are

$$Z \times \mathrm{PSL}_2(\mathbf{F}_\ell),\ 3.3^2.\mathrm{SL}_2(\mathbf{F}_3),\ Z \times A_5,\ Z \times \mathrm{PSL}_2(\mathbf{F}_7),\ \text{and } 3.A_6,$$

where Z is the center of $\mathrm{SL}_3(\mathbf{F}_\ell)$ (which has order 3 when $\ell \equiv 1(3)$ and is trivial otherwise). For the rest of this section set $H^0 := G^0 \cap \mathrm{SL}_3(\mathbf{F}_\ell)$.

Lemma 19. *The subgroup $\{1\} \times \mathrm{PSL}_2(\mathbf{F}_\ell)$ of $Z \times \mathrm{PSL}_2(\mathbf{F}_\ell)$ is an obstruction. Moreover, if H^0 is a Goursat-subgroup of $Z \times \mathrm{PSL}_2(\mathbf{F}_\ell)$, then H^0 is an obstruction if and only if $H^0 \simeq A_4$ or is dihedral.*

Proof. The symmetric square representation $\mathrm{Sym}^2 : \mathrm{SL}_2(\mathbf{F}_\ell) \longrightarrow \mathrm{SL}_3(\mathbf{F}_\ell)$ has kernel $\{\pm I\}$, whence the embedding $\mathrm{PSL}_2(\mathbf{F}_\ell) \hookrightarrow \mathrm{SL}_3(\mathbf{F}_\ell)$. If $g \in \mathrm{SL}_2(\mathbf{F}_\ell)$ has eigenvalues $\lambda^{\pm 1}$, then $\mathrm{Sym}^2(g)$ has eigenvalues $1, \lambda^{\pm 2}$. Since $\mathrm{PSL}_2(\mathbf{F}_\ell)$ is an irreducible subgroup of $\mathrm{SL}_3(\mathbf{F}_\ell)$, it is an obstruction.

Let H^0 be an obstruction given by the Goursat-tuple (H_1, H_2, H_3, ψ) with $H_1 \subset \mathrm{PSL}_2(\mathbf{F}_\ell)$ and $H_2 \subset Z$. Since Z is cyclic of order 3, there are three possibilities: $H_2 = H_3 = Z$, $H_2 = Z$ and $H_3 = 1$, or $H_2 = H_3 = 1$. The last case implies $H^0 \simeq H_1$, and we have just shown $\mathrm{PSL}_2(\mathbf{F}_\ell)$ is an obstruction.

If $H_2 = H_3 = Z$, then $H^0 \simeq H_1 \times Z$. Thus H^0 contains a non-trivial central element (which does not have 1 as an eigenvalue) hence cannot be an obstruction. It remains to assume that $H_2 = Z = \langle z \rangle$ and H_3 is trivial.

The subgroups of $\mathrm{PSL}_2(\mathbf{F}_\ell)$ are the projective images of the Borel and Cartan subgroups (and their normalizers) of $\mathrm{SL}_2(\mathbf{F}_\ell)$ and A_4, S_4 and A_5. The Cartan subgroups of $\mathrm{SL}_2(\mathbf{F}_\ell)$ are cyclic, while a suitable choice of basis puts a Borel subgroup into the form $\left\{\begin{pmatrix} \lambda^2 & * & * \\ 0 & 1 & * \\ 0 & 0 & \lambda^{-2} \end{pmatrix}\right\}$; neither can be an obstruction. On the other hand, the projective image of the normalizer of a Cartan subgroup is a dihedral group. The degree-3 representation in question is a direct sum of an irreducible degree-2 representation and a quadratic character, hence is an obstruction.

Of the groups A_4, S_4 and A_5, only A_4 has a $\mathbf{Z}/3$-quotient (necessary, since H_3 is trivial). It is easy to check that in this case H^0 is an obstruction isomorphic to A_4. $\square$

Lemma 20. *If $H^0 \subset Z \times \mathrm{PSL}_2(\mathbf{F}_7)$ is an obstruction, then $H^0 \simeq S_4$ or A_4.*

Proof. Let H^0 be given by the Goursat-tuple (H_1, H_2, H_3, ψ), with $H_1 \subset \mathrm{PSL}_2(\mathbf{F}_7)$, and $H_2 \subset Z$. There are three cases: $H_2 = H_3 = 1$, $H_2 = H_3 = Z$, or $H_2 = Z$ and $H_3 = 1$.

If H_2 is trivial, then H^0 is a subgroup of $\mathrm{PSL}_2(\mathbf{F}_7)$. According to Appendix A, the degree-3 character of $\mathrm{PSL}_2(\mathbf{F}_7)$ does not afford 1 as an eigenvalue on either of the conjugacy classes $7A$ or $7B$, hence $H^0 \subset S_4$. This representation of S_4 is irreducible and every element has 1 as an eigenvalue (Appendix A).

If $H_2 = H_3 = Z$, then $H^0 \simeq H_1 \times Z$ and thus $Z \subset H^0$. Since H^0 does not satisfy the fixed-point condition, it cannot be an obstruction. Finally, suppose that $H_2 = Z$ and H_3 is trivial. If $g \in \mathrm{PSL}_2(\mathbf{F}_7)$ has order 7, then it is impossible for $\psi(g)$ to have 1 as an eigenvalue ($\psi(g) = g, gz$, or gz^2, where $Z = \langle z \rangle$), hence $H_1 \subset S_4$. By assumption, H_1 has a $\mathbf{Z}/3$-quotient, hence is isomorphic to A_4 (otherwise H^0 is cyclic). Any surjective homomorphism $\psi : A_4 \longrightarrow Z$ gives rise to an obstruction H^0 isomorphic to A_4. $\square$

Lemma 21. *If $H^0 \subset Z \times A_5$ is an obstruction, then $H^0 \simeq A_5$ or A_4.*

Proof. As in the previous two lemmas, any $H^0 \subset Z \times A_5$ gives rise to a Goursat-tuple (H_1, H_2, H_3, ψ) with the only possibilities being $H_2 = H_3 = 1$ and $H_2 = H_2/H_3 = Z$. If H_2 is trivial, then $H^0 \simeq H_1 \subset A_5$. This representation of A_5 is irreducible and every element has 1 as an eigenvalue (Appendix A), hence A_5 is an obstruction.

If $H_2 = Z$ and $H_3 = 1$, then H_1 is a subgroup of A_5 with a $\mathbf{Z}/3$-quotient, hence is a subgroup of A_4. As in Lemma 20, any surjective homomorphism $\psi : A_4 \longrightarrow Z$ gives rise to an obstruction $H^0 \simeq A_4$. □

Next, recall from Section 3 that $3.3^2.\,\mathrm{SL}_2(\mathbf{F}_3)$ is the normalizer of the extra-special 3-group 3.3^2 (so that $\mathrm{SL}_2(\mathbf{F}_3)$ acts faithfully on 3^2 by conjugation). This group is maximal in $\mathrm{SL}_3(\mathbf{F}_\ell)$ if and only if $\ell \equiv 1(9)$; otherwise $3.3^2.Q_8$ is maximal.

Lemma 22. *Suppose $H^0 \subset 3.3^2.\,\mathrm{SL}_2(\mathbf{F}_3)$. Then H^0 cannot be an obstruction.*

Proof. According to [10, p. 149], 3.3^2 has presentation

$$3.3^2 = \langle x, y, z | x^3 = y^3 = z^3 = [x, z] = [y, z] = z^{-1}[x, y] = 1\rangle$$

(so z generates the center of 3.3^2), and there are exactly two inequivalent, absolutely irreducible, three-dimensional representations of 3.3^2, corresponding to the choices of a primitive 3^{rd} root of unity [10, prop. 4.6.3]. Choosing a basis and a primitive 3^{rd} root of unity ω we get a representation

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \text{ and } y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

and one checks that x and y generate all of 3.3^2. Furthermore, $\mathrm{SL}_2(\mathbf{F}_3)$ is generated by

$$A = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon^{-2} \end{pmatrix} \text{ and } B = (1-\omega)^{-1} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix},$$

where ϵ is a cube root of ω (so that $\ell \equiv 1(9)$). Neither A nor B have 1 as an eigenvalue, hence any subgroup H^0 of $3.3^2.\,\mathrm{SL}_2(\mathbf{F}_3)$ satisfying the fixed-point condition is a subgroup of of 3.3^2 which intersects the center trivially. Thus, H^0 is cyclic of order 3 and cannot be an obstruction. □

The last maximal irreducible subgroup of $\mathrm{SL}_3(\mathbf{F}_\ell)$ we have yet to analyze is $3.A_6$.

Lemma 23. *Let $H^0 \subset 3.A_6$ be an obstruction. Then H^0 is isomorphic to one of A_5, S_4, or A_4.*

Proof. Since $3.A_6$ contains the center of $\mathrm{SL}_3(\mathbf{F}_\ell)$, H^0 is a proper subgroup of $3.A_6$, and is therefore a subgroup of one of $3.3^2.4$, $Z \times A_5$, or $Z \times S_4$ ([4, p. 4]); Lemma 22 rules out $3.3^2.4$. According to Lemmas 20 and 21, H^0 is an obstruction if and only if $H^0 \simeq A_5$, S_4, or A_4. □

Together with the maximal subgroup structure of $\mathrm{SL}_3(\mathbf{F}_\ell)$ (cf. the beginning of this section), Lemmas 19-23 classify the maximal irreducible obstructions H^0 in $\mathrm{SL}_3(\mathbf{F}_\ell)$. We now consider the extension problem

$$1 \longrightarrow H^0 \longrightarrow G^0 \longrightarrow C \longrightarrow 1, \tag{4}$$

for any cyclic group $C \subset \mathbf{F}_\ell^\times$. If $\#C$ is even, then we define $\widetilde{C}$ to be the unique quotient of C of order $\#C/2$.

Lemma 24. *The solutions to the extension problem above are given in the following table.*

H^0	G^0
A_4	$A_4 \times C$ $S_4 \times \widetilde{C}$ *(if $\#C$ is even)*
S_4	$S_4 \times C$
A_5	$A_5 \times C$ $S_5 \times \widetilde{C}$ *(if $\#C$ is even)*
$\mathrm{PSL}_2(\mathbf{F}_\ell)$	$\mathrm{SL}_2(\mathbf{F}_\ell) \times C$ $\mathrm{SO}_3(\mathbf{F}_\ell) \times \widetilde{C}$ *(if $\#C$ is even)*

Proof. In each case H^0 has trivial center, so by [1, thm. 4.8] the G^0 containing H^0 are in one-to-one correspondence with homomorphisms $\phi : C \longrightarrow \mathrm{Out}(H^0)$. Except for S_4 (whose outer automorphism group is trivial) we have $\mathrm{Out}(H_0) \simeq \mathbf{Z}/2$.

When $H^0 \simeq S_4$, there is only one homomorphism $\phi : C \longrightarrow \mathrm{Out}(S_4)$, hence $G^0 \simeq S_4 \times C$. For the other cases, any cyclic group C admits at most two homomorphisms $\phi : C \longrightarrow \mathrm{Out}(H^0) \simeq \mathbf{Z}/2$. Therefore, the trivial extensions $H^0 \times C$ occur, along with the non-trivial extensions $H^0.2 \times \widetilde{C}$, where $H^0.2$ is a non-trivial extension of H^0 by $\mathbf{Z}/2$; there are isomorphisms $A_4.2 \simeq S_4$, $A_5.2 \simeq S_5$, and $\mathrm{PSL}_2(\mathbf{F}_\ell).2 \simeq \mathrm{SO}_3(\mathbf{F}_\ell)$ [10, prop. 2.9.1(ii)]. □

Proposition 8. *Let $H^0 \subset \mathrm{SL}_3(\mathbf{F}_\ell)$ be isomorphic to one of A_4, S_4, or $\mathrm{PSL}_2(\mathbf{F}_\ell)$. The irreducible GL_3-obstructions G^0 containing H^0 are S_4, A_5, and $\mathrm{SO}_3(\mathbf{F}_\ell)$.*

Proof. By Lemma 24, G^0 is isomorphic to either $H^0 \times C$ or $H^0.2 \times \widetilde{C}$, where C and $\widetilde{C}$ are cyclic groups of order dividing $\ell - 1$, and $H^0.2$ is a nontrivial extension of H^0.

We have seen that the degree-3 representations of A_4, S_4, and A_5 yield obstructions. However, S_5 has no irreducible, degree-3, $\mathbf{F}_\ell$-representations when $\ell \geq 7$, so we omit it from this analysis. The standard $\mathbf{F}_\ell$-representation of $\mathrm{SO}_3(\mathbf{F}_\ell)$ is irreducible, and by Lemma 31 $\mathrm{SO}_3(\mathbf{F}_\ell)$ satisfies the fixed-point condition, hence is an obstruction. It remains to show that S_4, A_5, and $\mathrm{SO}_3(\mathbf{F}_\ell)$ are *maximal* GL_3-obstructions. By Lemma 24, G^0 is isomorphic to one of $S_4 \times C$, $A_5 \times C$, or $\mathrm{SO}_3(\mathbf{F}_\ell) \times \widetilde{C}$; we will show C (resp. $\widetilde{C}$) is trivial.

Let c generate C (or $\widetilde{C}$). Since G^0 is a direct product, S_4 (resp. A_5, resp. $\mathrm{SO}_3(\mathbf{F}_\ell)$) commutes with c. The degree-3 representation of S_4 (resp. A_5, resp. $\mathrm{SO}_3(\mathbf{F}_\ell)$) is absolutely irreducible, so c must be scalar. Since G^0 satisfies the fixed-point condition, c, and hence C (resp. $\widetilde{C}$), must be trivial. □

In order to classify all irreducible GL_3-obstructions G^0, we need to revisit the case where H^0 is completely reducible and G^0/H^0 acts transitively on the irreducible factors, in accordance with Clifford's theorem. We can assume H^0 is diagonal and $G^0/H^0 \simeq \mathbf{Z}/3$ or S_3. Since G^0 satisfies the fixed-point condition, so does H^0.

Lemma 25. *Let H^0 be a diagonal subgroup of $\mathrm{SL}_3(\mathbf{F}_\ell)$. If both H^0 and G^0 are obstructions, then $H^0 \simeq \mathbf{Z}/2 \times \mathbf{Z}/2$ and $G^0 \simeq A_4$ or S_4.*

Proof. If H^0 is an obstruction, then $H^0 \simeq \mathbf{Z}/2 \times \mathbf{Z}/2$ by Lemma 3. When $G^0/H^0 \simeq \mathbf{Z}/3$ then $G^0 \simeq A_4$ and when $G^0/H^0 \simeq S_3$, then $G^0 \simeq S_4$. Both groups are obstructions. □

Lemma 26. *Let H^0 be a diagonal subgroup of $\mathrm{SL}_3(\mathbf{F}_\ell)$ and assume that both H^0 and G^0 satisfy the fixed-point condition. If H^0 is not an obstruction, then neither is G^0.*

Proof. Choose a basis so that

$$H^0 = \left\{ \begin{pmatrix} a_i & & \\ & a_i^{-1} & \\ & & 1 \end{pmatrix} \right\}, \ 1 \leq i \leq n.$$

Since G^0 acts transitively on the diagonal factors of H^0 and every element of G^0 is assumed to have 1 as an eigenvalue, it is easy to show that $n = 1$. Hence $G^0 \simeq \mathbf{Z}/3$ or S_3. Neither group has an irreducible degree-3 representation, contradicting the assumption that G^0 be irreducible. □

7.2. **From G^0 to G, Case I.** In this subsection we continue to assume that G^0 is an obstruction, and prove the following.

Proposition 9. *If $G^0 \subset \mathrm{GL}_3(\mathbf{F}_\ell)$ is an obstruction, then G is an obstruction.*

Proof. Any subgroup G of $\mathrm{GL}_3(\mathbf{F}_\ell).2$ can be written as a collection of 3×3 block matrices $\left\{ \begin{pmatrix} A_i & \\ & A_i^* \end{pmatrix}, \begin{pmatrix} & B_i \\ -B_i^* & \end{pmatrix} \right\}$, where $1 \leq i \leq n = \#G^0$. By assumption,

$$\begin{pmatrix} & B_j \\ -B_j^* & \end{pmatrix}^2 = \begin{pmatrix} -B_j B_j^* & \\ & -B_j^* B_j \end{pmatrix}$$

has 1 as an eigenvalue for all j, which is true if and only if $-B_jB_j^*$ has 1 as an eigenvalue for all j. Write $f_j(x)$ for the characteristic polynomial of $-B_jB_j^*$ and $g_j(x)$ for the characteristic polynomial of $\left(\begin{smallmatrix} & B_j \\ -B_j^* & \end{smallmatrix}\right)$ (so $g_j(1) = 0$ for all j). One checks that $g_j(x) = f_j(x^2)$, therefore $f_j(1) = 0$ for all j. Moreover, since the semisimplification of G^0 does not contain the trivial representation, neither does that of G, hence G is an obstruction. □

7.3. **From G^0 to G, Case II.** It remains to describe the groups $G \subset \mathrm{GL}_3(\mathbf{F}_\ell).2$ for which the semisimplification of G^0 contains the trivial representation. We will show that any such G is an obstruction. We give details in the special case where G^0 has the form:

$$\left\{ \left(\begin{smallmatrix} 1 & * & * & & & \\ 0 & * & * & & & \\ 0 & * & * & & & \\ & & & 1 & 0 & 0 \\ & & & * & * & * \\ & & & * & * & * \end{smallmatrix} \right) \right\}$$

and the non-trivial coset of G/G^0 consists of matrices $\left(\begin{smallmatrix} 0 & C_i \\ -C_i^* & 0 \end{smallmatrix}\right)$, $C_i \in \mathrm{GL}_3(\mathbf{F}_\ell)$.

First, any such G satisfies the fixed-point condition. If G were not an obstruction, then it would have to fix a line of the form $(\alpha, 0, 0, \beta, 0, 0)$, $\alpha, \beta \in \mathbf{F}_\ell^\times$. In terms of the non-trivial coset of G/G^0, this means each C_i and C_i^* must have the form $\left(\begin{smallmatrix} \alpha/\beta & * & * \\ 0 & * & * \\ 0 & * & * \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \beta/\alpha & * & * \\ 0 & * & * \\ 0 & * & * \end{smallmatrix}\right)$, respectively. However, this implies (since C_i^* is the dual of C_i) that $\alpha/\beta = -\alpha/\beta$, which contradicts the invertibility of the C_i.

8. The Field Extension Subgroups: $\mathrm{SL}_2(\mathbf{F}_{\ell^3}).3$ and $\mathrm{GU}_3(\mathbf{F}_{\ell^2}).2$

Let $\mathcal{G}$ denote either $\mathrm{SL}_2(\mathbf{F}_{\ell^3}).3$ or $\mathrm{GU}_3(\mathbf{F}_{\ell^2})$, $\mathcal{G}^0$ the subgroup of $\mathcal{G}$ isomorphic to $\mathrm{SL}_2(\mathbf{F}_{\ell^3})$ or $\mathrm{GU}_3(\mathbf{F}_{\ell^2})$, and $m = 3$ or 2, respectively. Define π relative to the split exact sequence

$$1 \longrightarrow \mathcal{G}^0 \longrightarrow \mathcal{G} \xrightarrow{\pi} \mathbf{Z}/m \longrightarrow 1.$$

Lemma 27. *Suppose $G \subset \mathcal{G}$ is such that $G \cap \ker \pi$ contains a copy of the trivial representation in its semisimplification. Then G contains a copy of the trivial representation in its semisimplification.*

Proof. By assumption, there exists a basis for $\mathbf{F}_{\ell^m}^{6/m}$ for which for which $G \cap \ker \pi \subset \mathcal{G}^0$ is in block upper-triangular form, with one of the blocks being trivial. The vector space isomorphism $\mathbf{F}_{\ell^m}^{6/m} \simeq \mathbf{F}_\ell^6$ preserves the block-upper triangular form of $G \cap \ker \pi$, hence its semisimplification (as a subgroup of $\mathrm{Sp}_6(\mathbf{F}_\ell)$) contains the trivial representation. By [10, (2.1.2)], the simple factors in $\mathrm{Sp}_6(\mathbf{F}_\ell)$ are preserved by the action of $\mathrm{Gal}(\mathbf{F}_{\ell^m}/\mathbf{F}_\ell)$. Since 1 is fixed by $\mathrm{Gal}(\mathbf{F}_{\ell^m}/\mathbf{F}_\ell)$, it follows that G contains a copy of the trivial representation in its semisimplification. □

Lemma 28. *Suppose every element of $G \subset \mathrm{SL}_2(\mathbf{F}_{\ell^3}).3$ has 1 as an eigenvalue. Then the semisimplification of G contains the trivial representation.*

Proof. Set $G^0 = G \cap \ker \pi$. By [12, p. I-2, ex. 1], there exists a basis for $\mathbf{F}_{\ell^3}^2$ such that $G^0 = \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$, *i.e.* G fixes a line L in $\mathbf{F}_{\ell^3}^2$. Under the isomorphism $\phi : \mathbf{F}_{\ell^3} \longrightarrow \mathbf{F}_\ell^3$ the line $\phi(L)$ is fixed by all of G^0, hence G^0 is not an obstruction, and by Lemma 27, neither is G. □

It remains to describe the obstructions $G \subset \mathrm{GU}_3(\mathbf{F}_{\ell^2}).2$. The geometric subgroups of $\mathrm{GU}_3(\mathbf{F}_{\ell^2})$ are:

$$\mathrm{GU}_2(\mathbf{F}_{\ell^2}) \times \mathbf{F}_{\ell^2}^\times \text{ [10, (4.1.4)]},\ \mathbf{F}_{\ell^6}^\times.3,\ \text{[10, (4.3.11)]},\ \mathbf{F}_{\ell^2}^\times \wr S_3,\ \text{[10, (4.2.9)], and } \mathrm{O}_3(\mathbf{F}_{\ell^2}) \text{ [10, (4.5.5)]},$$

while the exceptional subgroups of $\mathrm{SU}_3(\mathbf{F}_{\ell^2})$ are

$$\mathbf{Z}/3 \times \mathrm{PSL}_2(\mathbf{F}_7),\ 3.A_6,\ \text{and } 3.3^2.\mathrm{SL}_2(\mathbf{F}_3) \text{ [9, p. 200]}.$$

We begin by describing the obstructions for which $G = G^0 := G \cap \ker \pi \subset \mathrm{GU}_3(\mathbf{F}_{\ell^2})$.

Lemma 29. *If $G^0 \subset \mathrm{GU}_3(\mathbf{F}_{\ell^2})$ is an obstruction, then G^0 is isomorphic to one of $\mathbf{Z}/2 \times \mathbf{Z}/2$, S_3, D_4, A_4, S_4, $2.S_4$, D_n, $\mathrm{SO}_3(\mathbf{F}_\ell)$, or is an irreducible subgroup of $\mathrm{SO}_3(\mathbf{F}_{\ell^2})$.*

Proof. Any obstruction G^0 is a proper subgroup of $\mathrm{GU}_3(\mathbf{F}_{\ell^2})$, hence is contained in a maximal subgroup. If $G^0 \subset \mathrm{GU}_2(\mathbf{F}_{\ell^2}) \times \mathbf{F}_{\ell^2}^\times$, then Proposition 2 is easily adapted to this case to give $G^0 \simeq \mathbf{Z}/2 \times \mathbf{Z}/2$, D_4, $2.S_4$, or D_n.

When $G^0 \subset \mathbf{F}_{\ell^2}^\times \wr S_3$, set $H^0 := \ker \varpi \cap G^0$ relative to the split exact sequence:

$$1 \longrightarrow (\mathbf{F}_{\ell^2}^\times)^3 \longrightarrow \mathbf{F}_{\ell^2}^\times \wr S_3 \xrightarrow{\varpi} S_3 \longrightarrow 1.$$

If H^0 is an obstruction, then Lemma 3 shows $H^0 \simeq \mathbf{Z}/2 \times \mathbf{Z}/2$. Therefore, by Lemma 25 G^0 is isomorphic to either $\mathbf{Z}/2 \times \mathbf{Z}/2, A_4$, or S_4. If H^0 is not an obstruction, then one of the $\mathbf{F}_{\ell^2}^\times$-factors is trivial; choose a basis so that

$$H^0 = \left\{ \begin{pmatrix} a_i & & \\ & b_i & \\ & & 1 \end{pmatrix} \right\},\ i = 1, \ldots, n.$$

Using the fact that $G^0/H^0 \hookrightarrow S_3$ it is easy to show that $n = 1$ and $G^0 \simeq S_3$ (direct sum of the sign representation and the standard degree-2 representation).

It is not possible for $G^0 \subset \mathbf{F}_{\ell^6}^\times.3$ ($G^0 \cap \mathbf{F}_{\ell^6}^\times$ is a cyclic group which satisfies the fixed-point condition, hence it fixes a line in $\mathbf{F}_\ell^6$ - apply Lemma 27).

Finally, suppose $G^0 \subset \mathrm{O}_3(\mathbf{F}_{\ell^2})$. The maximal subgroups of $\mathrm{O}_3(\mathbf{F}_{\ell^2})$ are

$\mathrm{O}_1(\mathbf{F}_{\ell^2}) \times \mathrm{O}_2^\pm(\mathbf{F}_{\ell^2})$	[10, (4.1.5)]
$\mathrm{O}_1(\mathbf{F}_{\ell^2}) \wr S_3$	[10, (4.2.15)]
$\mathrm{O}_1(\mathbf{F}_{\ell^6}).3$	[10, (4.3.17)]
$\mathrm{O}_3(\mathbf{F}_\ell)$	[10, prop. 4.5.8],

where $\mathrm{O}_1(\mathbf{F}_{\ell^2}) = \mathrm{O}_1(\mathbf{F}_{\ell^6}) = \{\pm 1\}$. Each of $\mathrm{O}_1(\mathbf{F}_{\ell^2}) \times \mathrm{O}_2^\pm(\mathbf{F}_{\ell^2})$, $\mathrm{O}_1(\mathbf{F}_{\ell^2}) \wr S_3$, and $\mathrm{O}_1(\mathbf{F}_{\ell^6}).3$ are subgroups of $\mathbf{F}_{\ell^2}^\times \times \mathrm{GU}_2(\mathbf{F}_{\ell^2})$, $\mathbf{F}_{\ell^2}^\times \wr S_3$, and $\mathbf{F}_{\ell^6}^\times.3$ respectively, and have therefore already been analyzed. We are left with the case where G^0 is an irreducible subgroup of $\mathrm{O}_3(\mathbf{F}_{\ell^2})$.

By Clifford's theorem, $H^0 : G^0 \cap \mathrm{SO}_3(\mathbf{F}_{\ell^2})$ is irreducible. Every H^0 is an obstruction because $\mathrm{SO}_3(\mathbf{F}_{\ell^2})$ is (Lemma 31). Now apply the results of Lemma 24 to the exact sequence

$$1 \longrightarrow H^0 \longrightarrow G^0 \longrightarrow \langle -I \rangle \longrightarrow 1.$$

It follows that if G^0 is an irreducible obstruction, then $G^0 = H^0$. □

It remains to solve the extension problem

$$1 \longrightarrow G^0 \longrightarrow G \longrightarrow \mathrm{Gal}(\mathbf{F}_{\ell^2}/\mathbf{F}_\ell) \longrightarrow 1$$

for obstructions $G^0 \subset \mathrm{GU}_3(\mathbf{F}_{\ell^2})$.

Lemma 30. *Suppose $G^0 \subset \mathrm{GU}_3(\mathbf{F}_{\ell^2})$ is any obstruction on which $\mathrm{Gal}(\mathbf{F}_{\ell^2}/\mathbf{F}_\ell)$ acts non-trivially. Then G is an obstruction.*

Proof. Let $g \in G^0$ and suppose $gv = v$. Then $g^\sigma v = v$, for any $\sigma \in \mathrm{Gal}(\mathbf{F}_{\ell^2}/\mathbf{F}_\ell)$, by [10, (2.1.2)]. The semisimplification of G^0 does not contain the trivial representation, hence neither does G. Therefore G is an obstruction. □

9. Subgroups of $\mathrm{O}_3(\mathbf{F}_\ell) \otimes \mathrm{SL}_2(\mathbf{F}_\ell)$

Let A and B be finite groups equipped with finite dimensional representations $\rho_1 : A \longrightarrow \mathrm{GL}(V)$ and $\rho_2 : B \longrightarrow \mathrm{GL}(W)$. Denote by $A \otimes B$ the image of $\rho_1 \otimes \rho_2$ and observe that if $\{\lambda_1, \ldots, \lambda_m\}$ and $\{\mu_1, \ldots, \mu_m\}$ are the eigenvalues of $\rho_1(a)$ and $\rho_2(b)$ respectively, then $\{\lambda_1\mu_1, \ldots, \lambda_n\mu_m\}$ are the eigenvalues of $\rho_1(a) \otimes \rho_2(b)$.

The *orthogonal subgroup* $\mathrm{O}_3(\mathbf{F}_\ell)$ of $\mathrm{GL}_3(\mathbf{F}_\ell)$ is defined as the isometry group of a non-degenerate quadratic form $\mathbf{q}$; there exists a basis of $\mathbf{F}_\ell^3$ with respect to which $\mathbf{q}$ has the form $\left(\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$[10, prop. 2.5.3]. It is well known that $\det : \mathrm{O}_3(\mathbf{F}_\ell) \longrightarrow \{\pm 1\}$ is surjective, and we define $\ker \det := \mathrm{SO}_3(\mathbf{F}_\ell)$. According to [6, cor. 6.10], $\mathrm{SO}_3(\mathbf{F}_\ell)$ satisfies the fixed-point condition:

Lemma 31. [6, cor. 6.10] *Every element of $\mathrm{SO}_3(\mathbf{F}_\ell)$ has 1 as an eigenvalue.*

By forming the tensor product of the standard representations of $\mathrm{O}_3(\mathbf{F}_\ell)$ and $\mathrm{SL}_2(\mathbf{F}_\ell)$ we obtain the maximal subgroup $\mathrm{O}_3(\mathbf{F}_\ell) \otimes \mathrm{SL}_2(\mathbf{F}_\ell)$ of $\mathrm{Sp}_6(\mathbf{F}_\ell)$ [10, (4.4.14)]. If G is a subgroup of $\mathrm{O}_3(\mathbf{F}_\ell) \times \mathrm{SL}_2(\mathbf{F}_\ell)$, denote by $\mathcal{G}$ the associated subgroup of $\mathrm{O}_3(\mathbf{F}_\ell) \otimes \mathrm{SL}_2(\mathbf{F}_\ell)$.

Proposition 10. *Let $G \subset \mathrm{O}_3(\mathbf{F}_\ell) \times \mathrm{SL}_2(\mathbf{F}_\ell)$ be a direct product subgroup given by the Goursat-tuple (G_1, G_2, G_2, ψ). Then $\mathcal{G}$ is an obstruction if and only if $G_1 \subset \mathrm{O}_3(\mathbf{F}_\ell)$ is an obstruction and $G_2 \subset \mathrm{SL}_2(\mathbf{F}_\ell)$ is trivial or has order ℓ.*

Proof. Since G is a direct product, it follows that both G_1 and G_2 consist entirely of eigenvalue-1 elements, hence G_2 is trivial or has order ℓ. Thus $\mathcal{G}$ is an obstruction if and only if $G_1 \subset \mathrm{O}_3(\mathbf{F}_\ell)$ is an obstruction. □

For the remainder of this section G will be a Goursat-subgroup of $\mathrm{O}_3(\mathbf{F}_\ell) \times \mathrm{SL}_2(\mathbf{F}_\ell)$ with Goursat-tuple (G_1, G_2, G_3, ψ) and associated subgroup $\mathcal{G} \subset \mathrm{O}_3(\mathbf{F}_\ell) \otimes \mathrm{SL}_2(\mathbf{F}_\ell)$.

Lemma 32. *Let $\mathcal{G} \subset \mathrm{O}_3(\mathbf{F}_\ell) \otimes \mathrm{SL}_2(\mathbf{F}_\ell)$ be an obstruction. Then G_3 is either trivial or has order ℓ.*

Proof. Since $\{I\} \times G_3$ is a subgroup of G, each element of G_3 must have 1 as an eigenvalue. □

The maximal subgroups of $\mathrm{O}_3(\mathbf{F}_\ell)$ are

$\mathrm{O}_1(\mathbf{F}_{\ell^3}).3$	[10, (4.3.17)]
$\mathrm{O}_2^{\pm}(\mathbf{F}_\ell) \times \mathrm{O}_1(\mathbf{F}_\ell)$	[10, (4.1.5)]
$\mathrm{O}_1(\mathbf{F}_\ell) \wr S_3$	[10, (4.2.15)],

where $\mathrm{O}_1(\mathbf{F}_\ell) = \mathrm{O}_1(\mathbf{F}_{\ell^3}) = \{\pm 1\}$, and $\mathrm{O}_2^{\pm}(\mathbf{F}_\ell) \simeq D_{\ell \mp 1}$, the dihedral group of order $2(\ell \mp 1)$ [10, prop. 2.9.1].

Lemma 33. *If $G_1 = \mathrm{O}_1(\mathbf{F}_{\ell^3}).3$, then $\mathcal{G}$ is not an obstruction, and if $G_1 = \mathrm{O}_3(\mathbf{F}_\ell)$, then we get an obstruction $\mathcal{G} \simeq \mathrm{SO}_3(\mathbf{F}_\ell)$ by setting $G_2/G_3 = \{\pm I\}$ and $\ker\psi = \mathrm{SO}_3(\mathbf{F}_\ell)$.*

Proof. If $G_1 \simeq \mathrm{O}_1(\mathbf{F}_{\ell^3}).3$, then G_1 is cyclic, hence $\mathcal{G}$ is cyclic and therefore cannot be an obstruction. If $G_1 \simeq \mathrm{O}_3(\mathbf{F}_\ell)$, then $\psi : \mathrm{O}_3(\mathbf{F}_\ell) \longrightarrow \{\pm I\}$ defines a Goursat-subgroup G of $\mathrm{O}_3(\mathbf{F}_\ell) \times \mathrm{SL}_2(\mathbf{F}_\ell)$. The isomorphism $\mathcal{G} \simeq \mathrm{SO}_3(\mathbf{F}_\ell)$ follows since $\mathrm{O}_3(\mathbf{F}_\ell) = \mathrm{SO}_3(\mathbf{F}_\ell) \times \langle -I \rangle$ [10, (2.6.1)]. □

We divide the rest of this section into two cases based on the maximal subgroups $\mathrm{O}_2^{\pm}(\mathbf{F}_\ell) \times \mathrm{O}_1(\mathbf{F}_\ell)$ and $\mathrm{O}_1(\mathbf{F}_\ell) \wr S_3$ of $\mathrm{O}_3(\mathbf{F}_\ell)$ respectively.

9.1. **Case 1: $G_1 \subset \mathrm{O}_2^{\pm}(\mathbf{F}_\ell) \times \mathrm{O}_1(\mathbf{F}_\ell)$.** Until further notice set $G_1 \subset \mathrm{O}_2^{\pm}(\mathbf{F}_\ell) \times \mathrm{O}_1(\mathbf{F}_\ell)$ with Goursat-tuple (H_1, H_2, H_3, ϕ), where $H_1 \subset \mathrm{O}_2^{\pm}(\mathbf{F}_\ell)$, $H_2 \subset \mathrm{O}_1(\mathbf{F}_\ell) \simeq \{\pm 1\}$, and $\phi : H_1 \longrightarrow H_2/H_3$ is a surjective homomorphism. By [10, p. 44] there is an isomorphism $D_{\ell \mp 1} \longrightarrow \mathrm{O}_2^{\pm}(\mathbf{F}_\ell)$ defined by $r \mapsto \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$, and $s \mapsto \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. However, the following lemma (whose proof is omitted) shows that it is not possible for G_2 to be dihedral.

Lemma 34. *Let k be any field of characteristic $\neq 2$. Then $\mathrm{SL}_2(k)$ does not contain a dihedral group.*

There are three possibilities for H_2 and H_3, namely $H_2 = H_3 = 1$, $H_2 = H_3 = \{\pm 1\}$, or $H_2 = \{\pm 1\}$ and $H_3 = 1$. We treat these cases separately in the following lemmas.

Lemma 35. *Suppose G_1 is given by the Goursat-tuple (H_1, H_2, H_3, ϕ) with $H_2 = H_3 = 1$. Then $\mathcal{G}$ is an obstruction if and only if $G_1 = H_1$ is dihedral, $G_2/G_3 = \{\pm I\}$, and $\psi : G_1 \longrightarrow G_2/G_3$ is the natural homomorphism.*

Proof. If $H_2 = H_3 = 1$, then $G_1 \simeq H_1$. We have $H_1 \subset \mathrm{O}_2^{\pm}(\mathbf{F}_\ell) \simeq D_{\ell \mp 1}$, whence H_1 is either cyclic or dihedral.

If H_1 is cyclic, then G_2/G_3 is cyclic. Therefore, G_2 is either cyclic (when $G_3 = I$) or Borel (when G_3 has order ℓ). Let g be a generator for G_1 and set $\psi(g) = hG_3$. The fixed-point condition holds for $\mathcal{G}$ if and only if every element of $g \otimes hG_3$ has 1 as an eigenvalue, which is true if and only if $g \otimes h$ has 1 as an eigenvalue, by Lemma 32; it follows that the semisimplification of $\mathcal{G}$ contains the trivial representation.

If H_1 is dihedral, then $G_2/G_3 \simeq \{\pm I\}$ by Lemma 34. It follows that $\mathcal{G}$ is an obstruction. □

Proposition 11. *If $H_2 = H_3 = \{\pm 1\}$ (so $G_1 \simeq H_1 \times H_2$), or if $H_2 = \{\pm 1\}$ and H_3 is trivial, then $\mathcal{G}$ cannot be an obstruction.*

Proof. The group H_1 is either dihedral or cyclic; denote by $\langle h_1 \rangle$ its cyclic subgroup of index 2 or 1, respectively. First assume $H_2 = H_3 = \{\pm 1\}$. If $\mathcal{G}$ satisfies the fixed-point condition, then $\psi(h_1, -1)$ is non-trivial. Moreover, in order that the group structures on G_1 and G_2/G_3 be compatible with that on $\mathcal{G}$, it follows that $\ker \psi = H_1$. This forces $\mathcal{G}$ to fix a line in $\mathbf{F}_\ell^6$, and therefore fail to be an obstruction.

Next suppose H_1 is cyclic and H_3 is trivial so that G_1 is a cyclic group of even order. It follows that G_2/G_3 is cyclic and hence that $\mathcal{G}$ fixes a line. We are left with the case where H_1 is dihedral and $[H_1 : \ker \phi] = 2$, so that G_1 is dihedral. By Lemma 34, G_2/G_3 has order 2. The natural homomorphism $\psi : G_1 \longrightarrow G_2/G_3$ defines a group $\mathcal{G}$ which fixes a line in $\mathbf{F}_\ell^6$. This proves the proposition. □

9.2. **Case 2: $G_1 \subset \mathrm{O}_1(\mathbf{F}_\ell) \wr S_3$.** The maximal subgroup $\mathrm{O}_1(\mathbf{F}_\ell) \wr S_3$ of $\mathrm{O}_3(\mathbf{F}_\ell)$ is obtained through the natural S_3-action on

$$\mathrm{O}_1(\mathbf{F}_\ell)^3 \simeq \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix}, \ (\epsilon_i \in \{\pm 1\}),$$

where $\mathrm{O}_1(\mathbf{F}_\ell) \wr S_3$ is isomorphic to the trivial central extension $\langle -I \rangle \times S_4$ of (a degree-3 representation of) S_4.

Lemma 36. *Suppose $G \subset (\mathrm{O}_1(\mathbf{F}_\ell) \wr S_3) \times \mathrm{SL}_2(\mathbf{F}_\ell)$ gives rise to an obstruction $\mathcal{G} \subset \mathrm{O}_3(\mathbf{F}_\ell) \otimes \mathrm{SL}_2(\mathbf{F}_\ell)$. Then*

(a) $\ker \psi$ *consists entirely of elements having* 1 *as an eigenvalue,*
(b) G_2/G_3 *does not contain a subgroup isomorphic to* $\mathbf{Z}/2 \times \mathbf{Z}/2$ *or* S_3.

Proof. Part (a) follows from Lemma 35 and part (b) from Lemma 34. □

Our strategy to find all obstructions with $G_1 \subset \mathrm{O}_1(\mathbf{F}_\ell) \wr S_3$ is as follows. For each divisor d of 48, enumerate the subgroups G_1 of $\mathrm{O}_1(\mathbf{F}_\ell) \wr S_3$ of order d, then construct all possible homomorphisms $\psi : G_1 \longrightarrow G_2/G_3$. The computations are elementary but tedious, so we only provide the results. In the following table we list, for each $d|48$, the subgroups G_1 of $\mathrm{O}_1(\mathbf{F}_\ell) \wr S_3$ of order d, the normal subgroups K of G_1, and whether or not the group $\mathcal{G}$ (defined by $\ker \phi = K$) is an obstruction. If $\mathcal{G}$ fails to be an obstruction, it is usually because Lemma 36 is violated. If G_1 is cyclic, then so is $\mathcal{G}$ and therefore it cannot be an obstruction.

d	G_1	K	Obstruction?
48	$\mathrm{O}_1(\mathbf{F}_\ell) \wr S_3$	$\mathrm{O}_1(\mathbf{F}_\ell) \wr S_3$, $A_4 \times \{\pm I\}$, $\mathrm{O}_1(\mathbf{F}_\ell)^3$, A_4, $\#K = 1, 2$, or 4	No, Lemma 36
		S_4	Yes
24	$A_4 \times \{\pm I\}$	$A_4 \times \{\pm I\}$, $\mathrm{O}_1(\mathbf{F}_\ell)^3$, $\langle -I \rangle$, $\langle I \rangle$	No, Lemma 36
		A_4	Yes
		$\mathbf{Z}/2 \times \mathbf{Z}/2$	Yes
	S_4	$\mathbf{Z}/2 \times \mathbf{Z}/2$, $\langle I \rangle$	No Lemma 36
		A_4	Yes
16	$D_4 \times \mathbf{Z}/2$	$D_4 \times \mathbf{Z}/2$, $\mathbf{Z}/2 \times \mathbf{Z}/2 \times \mathbf{Z}/2$, $\mathbf{Z}/4 \times \mathbf{Z}/2$, $\#K = 1, 2$ or 4	No, Lemma 36
		D_4	Yes
12	D_6	D_6, $\mathbf{Z}/6$, $\mathbf{Z}/3$, $\langle -I \rangle$, $\{I\}$	No, Lemma 36
		S_3	Yes
	A_4	A_4, $\{I\}$	No, Lemma 36
		$\mathbf{Z}/2 \times \mathbf{Z}/2$	Yes
8	$\mathbf{Z}/4 \times \mathbf{Z}/2$	$\mathbf{Z}/4 \times \mathbf{Z}/2$, $\mathbf{Z}/4$, $\mathbf{Z}/2 \times \mathbf{Z}/2$	No, Lemma 36
		$\mathbf{Z}/4$	No, $\mathcal{G}$ is cyclic
	$\mathbf{Z}/2 \times \mathbf{Z}/2 \times \mathbf{Z}/2$	$\mathbf{Z}/2 \times \mathbf{Z}/2 \times \mathbf{Z}/2$, $\mathbf{Z}/2$, $\{I\}$	No, Lemma 36
		$\mathbf{Z}/2 \times \mathbf{Z}/2$	Yes
	D_4	$\mathbf{Z}/4$, $\#K = 1, 2$ or 8	No, Lemma 36
		$\mathbf{Z}/2 \times \mathbf{Z}/2$	Yes
6	S_3	$\{I\}$	No, Lemma 36
		$\mathbf{Z}/3$, S_3	Yes
4	$\mathbf{Z}/2 \times \mathbf{Z}/2$	$\{I\}$	No, Lemma 36
		$\mathbf{Z}/2 \times \mathbf{Z}/2$, $\mathbf{Z}/2$	Yes

To recap, we have:

Proposition 12. *Let $G_1 \subset \mathrm{O}_1(\mathbf{F}_\ell) \wr S_3$. Then $\mathcal{G}$ is an obstruction if and only if one of the following hold.*

G_1	G_2/G_3	$\mathcal{G}$
$\mathrm{O}_1(\mathbf{F}_\ell) \wr S_3$	$\{\pm I\}$	S_4
$A_4 \times \mathbf{Z}/2$	$\mathbf{Z}/6$	$A_4 \times \mathbf{Z}/2$
$A_4 \times \mathbf{Z}/2$	$\{\pm I\}$	A_4
S_4	$\{\pm I\}$	S_4
$D_4 \times \mathbf{Z}/2$	$\{\pm I\}$	D_4
D_6	$\{\pm I\}$	S_3
A_4	$\mathbf{Z}/3$	A_4
$\mathbf{Z}/2 \times \mathbf{Z}/2 \times \mathbf{Z}/2$	$\{\pm I\}$	$\mathbf{Z}/2 \times \mathbf{Z}/2$
D_4	$\{\pm I\}$	$\mathbf{Z}/2 \times \mathbf{Z}/2$
S_3	$\{\pm I\}$	S_3
$\mathbf{Z}/2 \times \mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2 \times \mathbf{Z}/2$

10. Exotic Subgroups

Let q be a power of ℓ and let $\pi : \mathrm{Sp}_6(\mathbf{F}_\ell) \longrightarrow \mathrm{PSp}_6(\mathbf{F}_\ell)$ be the natural projection. There are nine maximal subgroups (up to conjugation) of $\mathrm{PSp}_6(\mathbf{F}_q)$ that do not arise as stabilizers of vector space decompositions of $\mathbf{F}_q^6$ [9, p. 210]. We call these *exotic subgroups* (Type $\mathcal{S}$ in [9]) and list them in the following table:

Group	Conditions
$\mathrm{PSL}_2(\mathbf{F}_q)$	$q \geq 7$
$G_2(\mathbf{F}_q)$	q even
S_5	$q = \ell \equiv \pm 1(8)$
A_5	$q = \ell \equiv \pm 3(8)$
$\mathrm{PSL}_2(\mathbf{F}_7).a$	$\ell \notin \{2, 3, 7\}$, $\mathbf{F}_q = \mathbf{F}_\ell(\sqrt{2})$ $a = 2$ if $q \equiv \pm 1(16)$ $a = 1$ if $q \equiv \pm 3, \pm 5, \pm 7(16)$
$\mathrm{PSL}_2(\mathbf{F}_{13})$	$\ell \notin \{2, 13\}$, $\mathbf{F}_q = \mathbf{F}_\ell(\sqrt{13})$
A_7	$q = 9$
$\mathrm{PSU}_3(\mathbf{F}_9)$	$q = \ell \equiv \pm 1(12)$
J_2	q odd, $\mathbf{F}_q = \mathbf{F}_\ell(\sqrt{5})$

Now set $q = \ell \geq 7$ and suppose $\mathrm{im}(\pi \circ \overline{\rho_\ell}) = G$ is an exotic subgroup of $\mathrm{PSp}_6(\mathbf{F}_\ell)$ (so we disregard $G_2(q)$ and A_7). Since $2.G \subset \mathrm{Sp}_6(\mathbf{F}_\ell)$ is irreducible, it $2.G$ is an obstruction if and only if it satisfies the fixed-point condition. We will use the (ordinary) character table of $2.G$ to determine the characteristic polynomials (and therefore the eigenvalues) of its conjugacy classes; see Appendix A for lists of the characteristic polynomials.

10.1. **Case 1:** $G \subset \mathrm{PSL}_2(\mathbf{F}_\ell)$**.** Let $G = \mathrm{PSL}_2(\mathbf{F}_\ell)$ so that $\mathrm{Sym}^5(2.G) \simeq \mathrm{SL}_2(\mathbf{F}_\ell)$ defines an irreducible subgroup of $\mathrm{Sp}_6(\mathbf{F}_\ell)$ [9, p. 201]. We will show that no subgroup of $\mathrm{Sym}^5(2.G)$ is an obstruction.

Lemma 37. *Let $H \subset \mathrm{SL}_2(\mathbf{F}_\ell)$ and suppose $\mathrm{Sym}^5(H) \subset \mathrm{Sp}_6(\mathbf{F}_\ell)$ satisfies the fixed-point condition. Then the semisimplification of $\mathrm{Sym}^5(H)$ contains the trivial representation.*

Proof. If $h \in H$ has eigenvalues $\lambda^{\pm 1}$, then $\mathrm{Sym}^5(h)$ has eigenvalues $\lambda^{\pm 1}, \lambda^{\pm 3}$, and $\lambda^{\pm 5}$. It follows that either $\lambda = 1$ for all $h \in H$, or every element of H which does not have eigenvalues 1 has eigenvalues which are 3^{rd} or 5^{th} roots of unity. The former case cannot be an obstruction by [12, p. I-2, ex. 1], and in the latter case H is either Cartan (cyclic) or Borel. Neither can be an obstruction. □

10.2. **Case 2:** $G \subset A_5$ **and** S_5**.** The degree-6 characters χ_9^0 and χ_9^1 of $2.S_5$ do not have determinant 1, hence do not define subgroups of $\mathrm{Sp}_6(\mathbf{F}_\ell)$. On the other hand, the degree-6 character χ_9 of $2.A_5$ does have determinant 1.

Lemma 38. *Let $2.A_5 \subset \mathrm{Sp}_6(\mathbf{F}_\ell)$ be a non-trivial central extension of the exotic subgroup A_5 of $\mathrm{PSp}_6(\mathbf{F}_\ell)$ and let $H \subset 2.A_5$ be any subgroup. Then H is not an obstruction.*

Proof. The only non-trivial elements of $2.A_5$ which have 1 as an eigenvalue have orders 3 or 5. Therefore, if H satisfies the fixed-point condition, then it consists entirely of elements of orders 3 or 5. By the subgroup structure of $2.A_5$ this implies H is cyclic, and is therefore not an obstruction. □

10.3. **Case 3:** $G \subset \mathrm{PSL}_2(\mathbf{F}_7).a$ **or** $\mathrm{PSL}_2(\mathbf{F}_{13})$**.** Let G be any of the exotic subgroups $\mathrm{PSL}_2(\mathbf{F}_7).a$, or $\mathrm{PSL}_2(\mathbf{F}_{13})$ of $\mathrm{PSp}_6(\mathbf{F}_\ell)$. We will show that no subgroup H of $2.G$ can be an obstruction.

Suppose G is one of $\mathrm{PSL}_2(\mathbf{F}_{13})$ or $\mathrm{PSL}_2(\mathbf{F}_7)$ so that there are only two inequivalent central extensions $2.G$ (recall G is perfect). The trivial central extension is not a subgroup of $\mathrm{Sp}_6(\mathbf{F}_\ell)$ since G has no irreducible, degree-6 representation in characteristic ℓ, hence $2.G \simeq \mathrm{SL}_2(\mathbf{F}_{13})$ or $\mathrm{SL}_2(\mathbf{F}_7)$, respectively.

Proposition 13. *Let $H \subset \mathrm{SL}_2(\mathbf{F}_{13}) \subset \mathrm{Sp}_6(\mathbf{F}_\ell)$ be any subgroup. Then H is not an obstruction.*

Proof. We identify $\mathrm{SL}_2(\mathbf{F}_{13})$ with its degree-6 representation and suppose $H \subset \mathrm{SL}_2(\mathbf{F}_{13})$ satisfies the fixed-point condition. Combining the character data from Appendix A with the subgroup structure of $\mathrm{SL}_2(\mathbf{F}_{13})$ [4, p. 8], it follows that H is a subgroup of $2.A_4$ or $2.D_6$ which intersects the conjugacy class $1A_1$ trivially.

The only non-cyclic proper subgroup of $2.A_4$ is Q_8, but Q_8 intersects the class $1A_1$ non-trivially. Hence, H is cyclic and therefore cannot be an obstruction. A similar argument shows that no subgroup of $2.D_6$ can be an obstruction. □

Proposition 14. *Let $H \subset \mathrm{SL}_2(\mathbf{F}_7) \subset \mathrm{SL}_2(\mathbf{F}_\ell)$ be any subgroup. Then H is not an obstruction.*

Proof. According to Appendix A, no degree-6 character of $\mathrm{SL}_2(\mathbf{F}_7)$ affords 1 as an eigenvalue on the conjugacy classes $7A_0$, $7A_1$, $7B_0$, or $7B_1$. Hence if H satisfies the fixed-point condition, then $7|[\mathrm{SL}_2(\mathbf{F}_7) : H]$ and H intersects the center of $\mathrm{SL}_2(\mathbf{F}_7)$ trivially. Accordingly, $\pi(H) \simeq H$ is a subgroup of S_4. There are no non-cyclic subgroups of S_4 which can be embedded in $\mathrm{SL}_2(\mathbf{F}_7)$ (Lemma 34), hence any H satisfying the fixed-point condition must be cyclic of order 3 (a cyclic subgroup of order 4 contains the center) and therefore not an obstruction. □

If $G \simeq \mathrm{PSL}_2(\mathbf{F}_7).2$, and $H \subset 2.G$ is any subgroup, define $H^0 := H \cap \ker \phi \subset \mathrm{SL}_2(\mathbf{F}_7)$ relative to the short exact sequence:

$$1 \longrightarrow \mathrm{SL}_2(\mathbf{F}_7) \longrightarrow \mathrm{SL}_2(\mathbf{F}_7).2 \xrightarrow{\phi} \mathbf{Z}/2 \longrightarrow 1 .$$

Lemma 39. *Let $H \subset \mathrm{SL}_2(\mathbf{F}_7).2$ be any subgroup. Then H is not an obstruction.*

Proof. Suppose H (and therefore H^0) satisfies the fixed-point assumption. By Proposition 14, H^0 is cyclic of order 3. It suffices to assume $H \simeq S_3$; denote by χ the associated degree-6 character of H. By Maschke's theorem, χ is completely reducible, and the following table shows that χ is a direct sum of two copies of the degree-2 representation, one copy of the sign representation, and one copy of the trivial representation, which proves the lemma.

Rep. of S_3	Characteristic Polynomial of $1A_0$	Characteristic Polynomial of $3A_0$	Characteristic Polynomial of $2B_0$
χ	$(x-1)^6$	$(x-1)^2(x^2+x+1)^2$	$(x-1)^3(x+1)^3$
trivial	$(x-1)$	$(x-1)$	$(x-1)$
alternating	$(x-1)$	$(x-1)$	$(x+1)$
degree-2	$(x-1)^2$	(x^2+x+1)	(x^2-1)

□

10.4. **Case 4: $G \subset \mathrm{PSU}_3(\mathbf{F}_9)$.** The preimage of $\mathrm{PSU}_3(\mathbf{F}_9)$ in $\mathrm{Sp}_6(\mathbf{F}_\ell)$ is $\mathrm{PSU}_3(\mathbf{F}_9)$ [4, p. 14], and any obstruction $G \subset \mathrm{PSU}_3(\mathbf{F}_9)$ must be a proper subgroup since (for example) the conjugacy class $3A$ does not afford 1 as an eigenvalue. The maximal subgroups of $\mathrm{PSU}_3(\mathbf{F}_9)$ are

$$3^{1+2}_+ : 8,\ \mathrm{PSL}_2(\mathbf{F}_7),\ 4^2 : S_3,\ \text{and}\ 4 \cdot S_4.$$

None of these groups satisfy the fixed-point condition either, so we investigate the next level of maximal subgroups. With the aid of Magma, we can write down the maximal subgroup lattice and then check which groups satisfy the fixed-point condition using the characteristic polynomials in Appendix A. We briefly sketch the results.

Any (non-cyclic) $G \subset 3^{1+2}_+ : 8$ which satisfies the fixed-point condition cannot meet the conjugacy class $3A$, nor can it have an element of order 6 or 8. This means G is a non-cyclic proper subgroup of $3^2 : 2$, hence is a subgroup of S_3 or 3^2. Neither group can be an obstruction.

Next suppose G is a subgroup of $\mathrm{PSL}_2(\mathbf{F}_7)$ which satisfies the fixed-point condition. Any such G does not meet $7A$, hence is a subgroup of S_4. Using the methods of Lemma 14, it can be shown that the six-dimensional representations of S_4 in question contain the trivial representation in their semisimplifications.

The group $4.S_4$ has three maximal subgroups, of orders 24, 32, and 48, respectively. The group of order 24 intersects $8A$ non-trivially,and each of its maximal subgroups is cyclic. Therefore this group of order 24 does not lead to an obstruction. The maximal subgroup of order 32 intersects $8A$ non-trivially, and it can be shown any non-cyclic proper subgroup which satisfies the fixed-point assumption necessarily contains a copy of the trivial representation. Similarly, one can show that no subgroup of $4^2 : S_3$ can be an obstruction.

10.5. **Case 5:** $G \subset J_2$**.** The largest exotic subgroup of $\mathrm{PSp}_6(\mathbf{F}_\ell)$ is the Hall-Janko group J_2; it's preimage in $\mathrm{Sp}_6(\mathbf{F}_\ell)$ is a non-trivial double-cover $2.J_2$. One of the assumptions of Theorem 1 is that $\operatorname{im} \pi \circ \overline{\rho_\ell}$ is not a proper subgroup J_2. One could (in principle) remove this assumption by searching the subgroup lattice of J_2 for obstructions; a recursive approach is outlined here.

Any degree-6 irreducible representation of $2.J_2$ does not afford 1 as an eigenvalue on conjugacy class $3A$, hence any subgroup G satisfying the fixed-point assumption must be contained in a maximal subgroup of $2.J_2$. Moreover, such a group G can only contain elements from the conjugacy classes $1A_0$, $2A_0$, $2A_1$, $2B$, $3B_0$, $4A_0$, $6A_1$, $8A_0$, and $12A_1$. Using the recursive procedure outlined in this paper, one could determine all subgroups of $2.J_2$ which satisfy the fixed-point condition. To decide whether or not the given group is an obstruction, apply Maschke's theorem and compare the characteristic polynomials of the irreducible constituents to the given characteristic polynomials; this will determine whether or not the group is an obstruction.

Appendix A. Characteristic Polynomials

In this appendix we record the characteristic polynomials which are referred to in the main text and briefly describe how they are obtained.

Let $\mathcal{C}$ be a conjugacy class of a finite group G, and let χ be an irreducible character of G. One can extract the coefficients of the characteristic polynomial of $\mathcal{C}$ from the values of χ by expressing the elementary symmetric polynomials in terms of the power-sum polynomials (or *Newton* polynomials) [11, p. 15]; the first six are:

$$e_1 = \chi(\mathcal{C}),\ e_2 = \frac{\chi(\mathcal{C})^2 - \chi(\mathcal{C}^2)}{2},\ e_3 = \frac{\chi(\mathcal{C})^3 - 3\chi(\mathcal{C})\chi(\mathcal{C}^2) + 2\chi(\mathcal{C}^3)}{6}$$

$$e_4 = \frac{\chi(\mathcal{C})^4 - 6\chi(\mathcal{C}^2)\chi(\mathcal{C})^2 + 3\chi(\mathcal{C}^2) + 8\chi(\mathcal{C}^3)\chi(\mathcal{C}) - 6\chi(\mathcal{C}^4)}{24}$$

$$\begin{aligned} e_5 &= \frac{\chi(\mathcal{C})^5 - 10\chi(\mathcal{C}^2)\chi(\mathcal{C})^3 + 15\chi(\mathcal{C}^2)^2\chi(\mathcal{C}) - 20\chi(\mathcal{C}^3)\chi(\mathcal{C}^2) + 20\chi(\mathcal{C}^3)\chi(\mathcal{C})^2}{120} \\ &+ \frac{-30\chi(\mathcal{C}^4)\chi(\mathcal{C}) + 24\chi(\mathcal{C}^5)}{120} \\ e_6 &= \frac{\chi(\mathcal{C})^6 - 15\chi(\mathcal{C}^2)\chi(\mathcal{C})^4 + 45\chi(\mathcal{C}^2)^2\chi(\mathcal{C})^2 - 15\chi(\mathcal{C}^2)^3 + 40\chi(\mathcal{C}^3)\chi(\mathcal{C})^3}{720} \\ &+ \frac{-120\chi(\mathcal{C}^3)\chi(\mathcal{C}^2)\chi(\mathcal{C}) + 40\chi(\mathcal{C}^3) - 90\chi(\mathcal{C}^4)\chi(\mathcal{C})^2 + 90\chi(\mathcal{C}^4)\chi(\mathcal{C}^2)}{720} \\ &+ \frac{144\chi(\mathcal{C}^5)\chi(\mathcal{C}) - 120\chi(\mathcal{C}^6)}{720} \end{aligned}$$

One of the key assumptions for our classification of obstructions in $\mathrm{Sp}_6(\mathbf{F}_\ell)$ was that $\ell \geq 7$. This allows us to avoid modular representations where the characteristic of the field divides the order of the group. Moreover, the $\mathbf{F}_\ell$-representations that we encounter are the "reductions mod ℓ" of the ordinary $\mathbf{C}$-valued representations. Since $\mathbf{F}_\ell$ is not algebraically closed, there may be congruence conditions imposed on ℓ in order that such a representation be $\mathbf{F}_\ell$-rational. For example, $7 | \#J_2$, yet J_2 is not a subgroup of $\mathrm{PSp}_6(\mathbf{F}_7)$ since 5 is not a square mod 7.

The obstructions G for which $\ell \mid \#G$ involve the groups $\mathrm{Sym}^2 \mathrm{SL}_2(\mathbf{F}_\ell)$, $\mathrm{SO}_3(\mathbf{F}_\ell)$, and $\mathrm{SO}_3(\mathbf{F}_{\ell^2})$. In these cases, we only use the fact that the natural $\mathbf{F}_q$-representation of $\mathrm{SO}_3(\mathbf{F}_q)$ and the symmetric power representations of $\mathrm{SL}_2(\mathbf{F}_\ell)$ are irreducible. To study their subgroups, we appealed to the classification of the subgroups of the finite linear groups [10].

We now record the characteristic polynomials, using Atlas notation where appropriate.

A_4

	1	(123)	(132)	(12)(34)
χ (degree-3)	$(x-1)^3$	$(x-1)(x^2+x+1)$	$(x-1)(x^2+x+1)$	$(x-1)(x+1)^2$

$2.A_4$

	$1A_0$	$1A_1$	$2A_0$	$3A_0$	$3A_1$	$3B_0$	$3B_1$
χ_1 (degree 2)	$(x-1)^2$	$(x+1)^2$	x^2+1	x^2-x+1	x^2+x+1	x^2-x+1	x^2+x+1
χ_2 (degree 2)	$(x-1)^2$	$(x+1)^2$	x^2+1	x^2-z6^2x+1	$x^2-z6x+1$	x^2-z6^4x+1	x^2-z6^5x+1
χ_3 (degree 2)	$(x-1)^2$	$(x+1)^2$	x^2+1	x^2-z6^4x+1	x^2-z6^5x+1	x^2-z6^2x+1	$x^2-z6x+1$

S_4

	1	(12)	(123)	(1234)	(12)(34)
χ_1 (degree 2)	$(x-1)^2$	x^2-1	x^2+x+1	x^2-1	x^2-2x+1
χ_2 (degree 3)	$(x-1)^3$	$(x-1)^2(x+1)$	$(x-1)(x^2+x+1)$	$(x+1)(x^2+1)$	$(x-1)(x^2+x+1)$
χ_3 (degree 3)	$(x-1)^3$	$(x+1)^2(x-1)$	$(x-1)(x^2+x+1)$	$(x-1)(x^2+1)$	$(x-1)(x^2+x+1)$

$2.S_4$

	1	2	3	4	5	6	7	8
χ_1 (degree 2)	$(x-1)^2$	$(x+1)^2$	x^2+x+1	x^2-x+1	x^2+1	$x^2-r2x+1$	$x^2+r2x+1$	x^2+1
χ_2 (degree 2)	$(x-1)^2$	$(x+1)^2$	x^2+x+1	x^2-x+1	x^2+1	$x^2+r2x+1$	$x^2-r2x+1$	x^2+1

$2.A_5$

	χ_6	χ_7	χ_9
$1A_0$	$(x-1)^2$	$(x-1)^2$	$(x-1)^6$
$1A_1$	$(x+1)^2$	$(x+1)^2$	$(x+1)^6$
$2A_0$	x^2+1	x^2+1	$(x^2+1)^3$
$3A_0$	x^2+x+1	x^2+x+1	$(x-1)^2(x^2+x+1)^2$
$3A_1$	x^2-x+1	x^2-x+1	$(x+1)^2(x^2-x+1)^2$
$5A_0$	$x^2-b5x+1$	x^2-b5^*x+1	$(x-1)^2(x^4+x^3+x^2+x+1)$
$5A_1$	$x^2+b5x+1$	x^2+b5^*x+1	$(x+1)^2(x^4-x^3+x^2-x+1)$
$5B_0$	x^2-b5^*x+1	$x^2-b5x+1$	$(x-1)^2(x^4+x^3+x^2+x+1)$
$5B_1$	x^2+b5^*x+1	$x^2+b5x+1$	$(x+1)^2(x^4-x^3+x^2-x+1)$

S_5

	$\chi_{2,3}$
$1A$	$(x-1)^6$
$2A$	$(x-1)^2(x+1)^4$
$3A$	$(x-1)^2(x^2+x+1)^2$
$5AB$	$(x-1)^2(x^4+x^3+x^2+x+1)$
$2B$	$(x-1)^3(x+1)^3$
$4A$	$(x-1)(x+1)(x^2+1)^2$
$6B$	$(x-1)(x+1)(x^2+x+1)(x^2-x+1)$

$2.S_5$

	χ_9^0	χ_9^1
$1A_0$	$(x-1)^6$	$(x-1)^6$
$1A_1$	$(x+1)^6$	$(x+1)^6$
$2A_0$	$(x^2+x+1)^3$	$(x^2+1)^3$
$3A_0$	$(x-1)^2(x^2+x+1)^2$	$(x-1)^2(x^2+x+1)^2$
$3A_1$	x^2-x+1	$(x+1)^2(x^2-x+1)^2$
$5AB_0$	$(x-1)^2(x^4+x^3+x^2+x+1)$	$(x-1)^2(x^4+x^3+x^2+x+1)$
$5AB_1$	$(x+1)^2(x^4-x^3+x^2-x+1)$	$(x+1)^2(x^4-x^3+x^2-x+1)$
$2B_0$	$(x-1)^3(x+1)^3$	$(x-1)^3(x+1)^3$
$4A_0$	$x^6-i2x^5-x^4+x^2-i2x-1$	$x^6+i2x^5-x^4+x^2+i2x-1$
$4A_1$	$x^6+i\sqrt{2}x^5-x^4+x^2+i2x-1$	$x^6-i\sqrt{2}x^5-x^4+x^2-i2x-1$
$6B_0$	$(x-1)(x+1)(x^2+x+1)(x^2-x+1)$	$(x-1)(x+1)(x^2+x+1)(x^2-x+1)$
$6B_1$	$(x-1)(x+1)(x^2+x+1)(x^2-x+1)$	$(x-1)(x+1)(x^2+x+1)(x^2-x+1)$

$2.A_6$

	χ_8	χ_9
$1A_0$	$(x-1)^4$	$(x-1)^2$
$1A_1$	$(x+1)^4$	$(x+1)^2$
$2A_0$	$(x^2+1)^2$	$(x^2+1)^2$
$3A_0$	$(x^2+x+1)^2$	$(x^2+x+1)^2$
$3A_1$	$(x^2-x+1)(x+1)^2$	$(x^2-x+1)^2$
$3B_0$	$(x^2+x+1)^2$	$(x^2+x+1)^2$
$3B_1$	$(x^2-x+1)^2$	$(x^2-x+1)(x+1)^2$
$4A_0$	x^4+1	x^4+1
$4A_1$	x^4+1	x^4+1
$5A_0$	$x^4+x^3+x^2+x+1$	$x^4+x^3+x^2+x+1$
$5A_1$	$x^4-x^3+x^2-x+1$	$x^4-x^3+x^2-x+1$
$5B_0$	$x^4+x^3+x^2+x+1$	$x^4+x^3+x^2+x+1$
$5B_1$	$x^4-x^3+x^2-x+1$	$x^4-x^3+x^2-x+1$

$2.S_6$

	χ_8^0	χ_8^1	χ_9^0	χ_9^1
$2B_0$	$(x-1)^2(x+1)^2$	$(x-1)^2(x+1)^2$	$(x-1)^2(x+1)^2$	$(x-1)^2(x+1)^2$
$2C_0$	$(x^2+1)^2$	$(x^2+1)^2$	$(x^2+1)^2$	$(x^2+1)^2$
$4B_0$	x^4+1	x^4+1	x^4+1	x^4+1
$6A_0$	$(x^2-x+1)(x^2+x+1)$	$(x^2-x+1)(x^2+x+1)$	$x^4-i3x^3-2x^2+i3x+1$	$x^4+i3x^3-2x^2-i3x+1$
$6A_1$	$(x^2-x+1)(x^2+x+1)$	$(x^2-x+1)(x^2+x+1)$	$x^4+i3x^3-2x^2-i3x+1$	$x^4-i3x^3-2x^2+i3x+1$
$6B_0$	$x^4-r3x^3+2x^2-r3x+1$	$x^4+r3x^3+2x^2+r3x+1$	x^4-x^2+1	x^4-x^2+1
$6B_1$	$x^4+r3x^3+2x^2+r3x+1$	$x^4-r3x^3+2x^2-r3x+1$	x^4-x^2+1	x^4-x^2+1

$\mathrm{PSL}_2(\mathbf{F}_7)$

	χ_2	χ_3	χ_4
$1A$	$(x-1)^3$	$(x-1)^3$	$(x-1)^6$
$2A$	$(x-1)(x+1)^2$	$(x-1)(x+1)^2$	$(x-1)^4(x+1)^2$
$3A$	$(x-1)(x^2+x+1)$	$(x-1)(x^2+x+1)$	$(x-1)^2(x^2+x+1)^2$
$4A$	$(x-1)(x^2+1)$	$(x-1)(x^2+1)$	$(x-1)^2(x+1)^2(x^2+1)$
$7A$	$x^3-b7x^2+b7^*x-1$	$x^3+b7x^2-b7^*x-1$	$x^6+x^5+x^4+x^3+x^2+x+1$
$7B$	$x^3+b7x^2-b7^*x-1$	$x^3-b7x^2+b7^*x-1$	$x^6+x^5+x^4+x^3+x^2+x+1$

Acknowledgements. We would like to thank Nigel Boston, Farshid Hajir, Jim Humphreys, and Arunas Rudvalis for helpful discussions. We especially thank Siman Wong for suggesting this problem and his many helpful comments.

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