Pentahedral Volume, Chaos, and Quantum Gravity

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Introduction

I show that chaotic classical dynamics associated to the volume of discrete grains of space leads to quantal spectra that are gapped between zero and nonzero volume. This strengthens the connection between spectral discreteness in the quantum geometry of gravity and tame ultraviolet behavior. I complete a detailed analysis of the geometry of a pentahedron, providing new insights into the volume operator and evidence of classical chaos in the dynamics it generates. These results reveal an unexplored realm of application for chaos in quantum gravity. [1]

Dynamical polyhedra

Minkowski proved in 1897 that the shape of a convex polyhedron is determined by its area vectors $\{\vec{A}_\ell\}$:

$|\vec{A}|$ = area of face; $\hat{A}$ = normal to face,

satisfying closure,

$$\vec{A}_1 + \cdots + \vec{A}_N = 0.$$

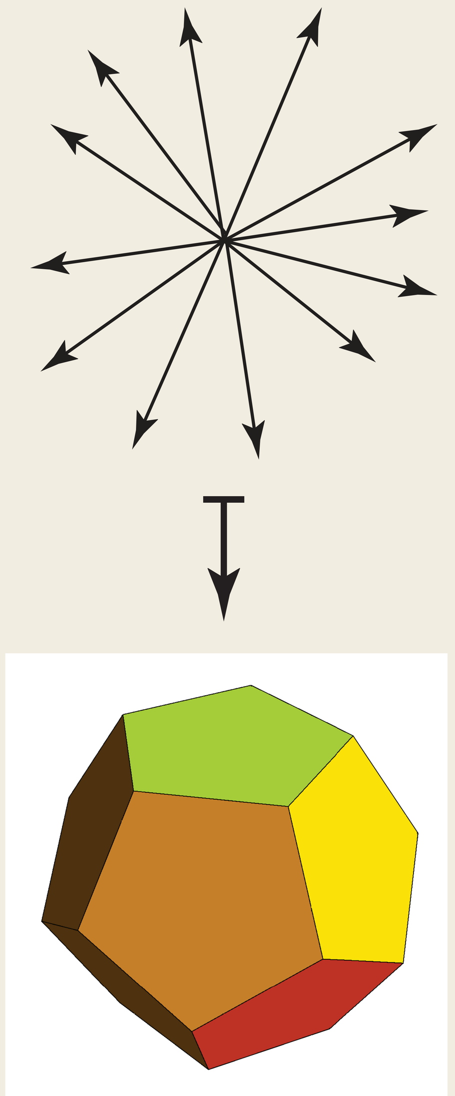
His proof was not constructive, and given the area vectors it is a **challenge** to build a polyhedron.

If we interpret area vectors physically as angular momenta, then polyhedra become dynamical systems. With the usual Poisson brackets

$$\{A^i_\ell, A^j_\ell\} = \epsilon^{ij}{}_k A^k_\ell$$

any function of the area vectors can be taken as a Hamiltonian; here we study the pentahedral volume

$$H = V_{\text{pent}}(\vec{A}_\ell).$$



Pentahedral volume

A pentahedron can be continued to a tetrahedron by extending 3 of its faces. The face areas of the pentahedron are scaled by α, β, and γ. These scalings can be found from the tetrahedral closure

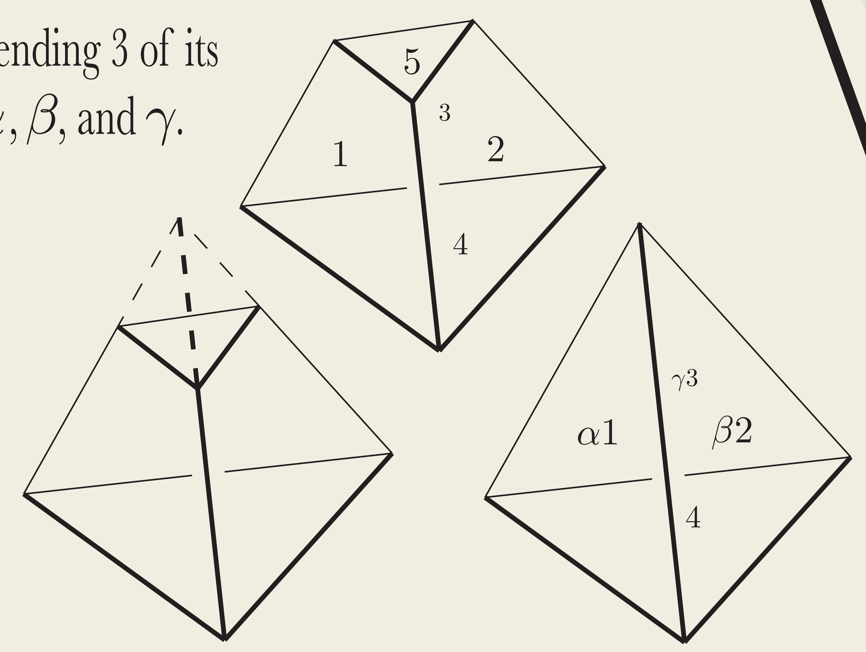
$$\alpha\vec{A}_1 + \beta\vec{A}_2 + \gamma\vec{A}_3 + \vec{A}_4 = 0,$$

for example,

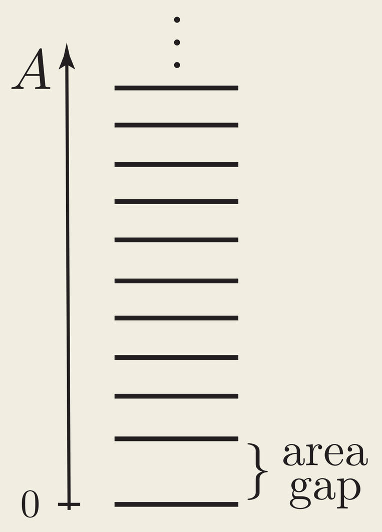
$$\gamma = -\frac{\vec{A}_4\cdot(\vec{A}_1\times\vec{A}_2)}{\vec{A}_3\cdot(\vec{A}_1\times\vec{A}_2)}.$$

The pentahedron's volume can now be expressed as a function of the known tetrahedral volumes

$$V_{\text{pent}} = \frac{\sqrt{2}}{3}\left(\sqrt{\alpha\beta\gamma} - \sqrt{(\alpha-1)(\beta-1)(\gamma-1)}\right)\sqrt{\vec{A}_1\cdot(\vec{A}_2\times\vec{A}_3)}$$



Spectral gaps in quantum gravity

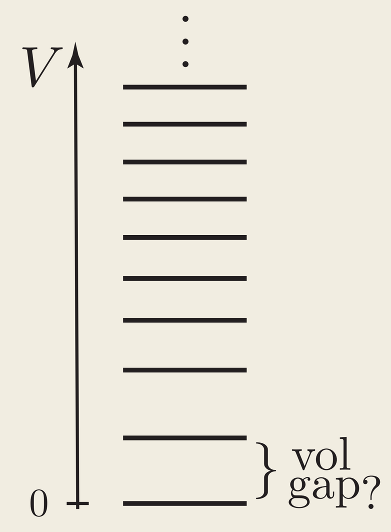


In loop gravity both area and volume become quantum operators. The area spectrum has long been known to be gapped.

- Is there a volume gap?

A discrete volume spectrum seems to imply yes, but doubts have been raised; e.g., for an equi-area pentahedron the number of volume states grows as A^2 but the allowed range of volume grows more slowly, $V_{\max} \sim A^{\frac{3}{2}}$. So, more and more volume states are being crowded into the allowed range. The core question then is:

- How robust is a volume gap?

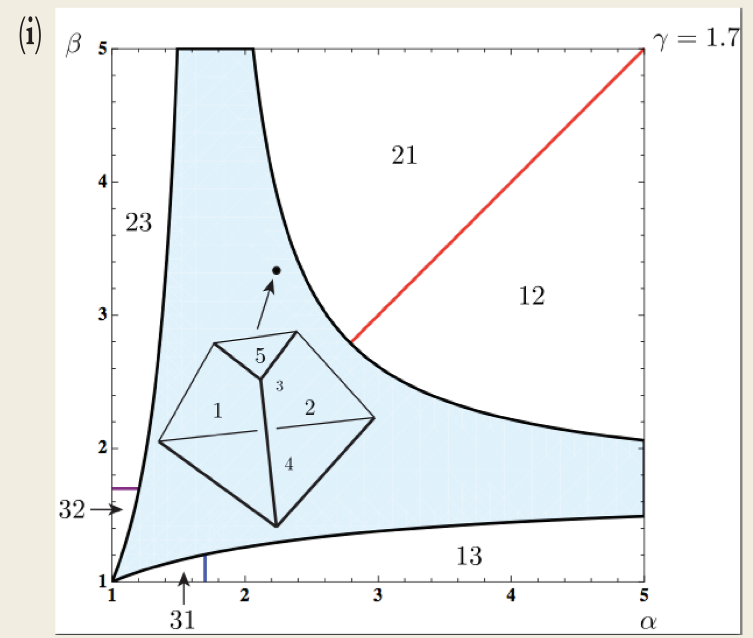


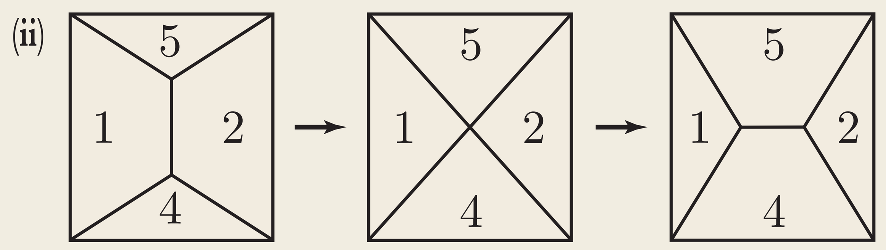
Pentahedral phase diagram

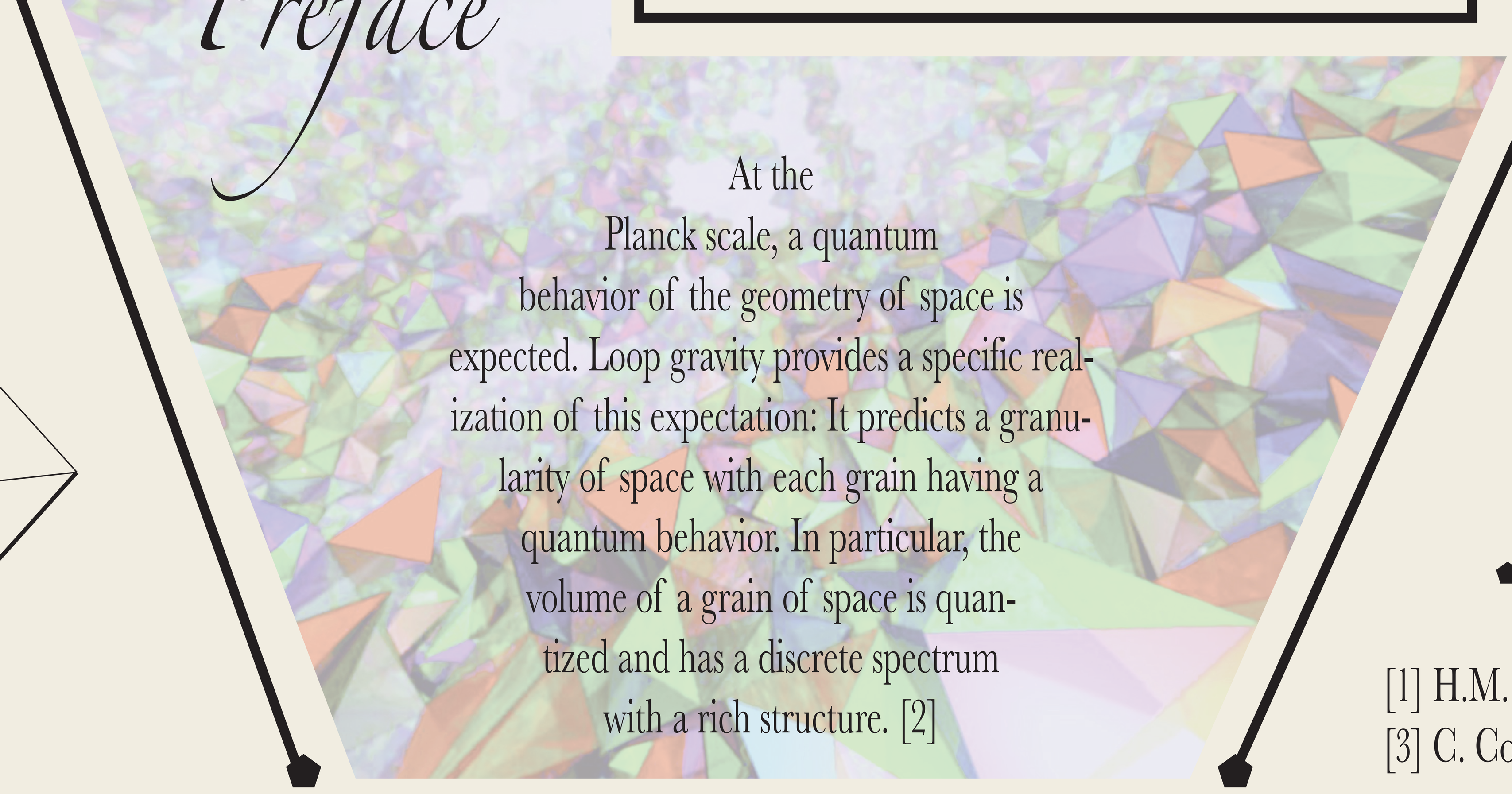
Call it a 54-pentahedron if the two opposite triangular faces are face 5 and face 4; this determines the adjacency of all the faces. Remarkably, the α, β, and γ parameters completely determine the adjacency of a given set of 5 area vectors. For example, the tetrahedral continuation of a 53-pentahedron leads to a γ' scaling,

$$\gamma' = -\frac{\vec{A}_3\cdot(\vec{A}_1\times\vec{A}_2)}{\vec{A}_4\cdot(\vec{A}_1\times\vec{A}_2)} = \frac{1}{\gamma}$$

and hence, because α, β, and $\gamma > 1$, the 54- and 53-pentahedra can't both be constructable. Continuing in this manner a pentahedral phase diagram can be built [see (**i**)]. This allows you to **solve** the problem of building a pentahedron given its area vectors [1]. The different adjacency classes are connected to each other by Pachner moves [see (**ii**) for a schematic].





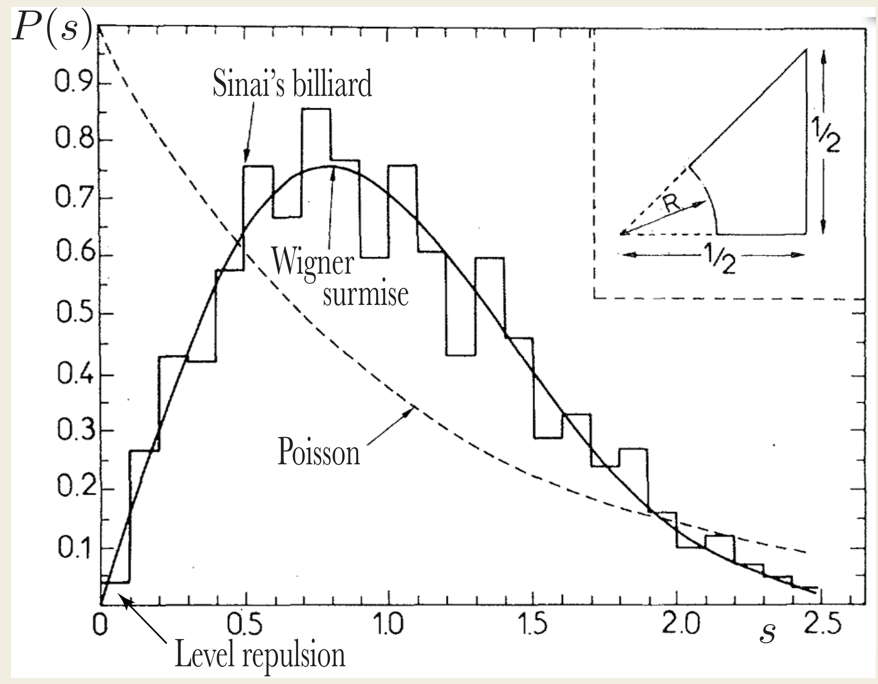


Quantum chaos

Chaotic quantum systems generically exhibit level repulsion: the probability $P(s)$ of a level spacing s rapidly goes to zero with the level spacing.

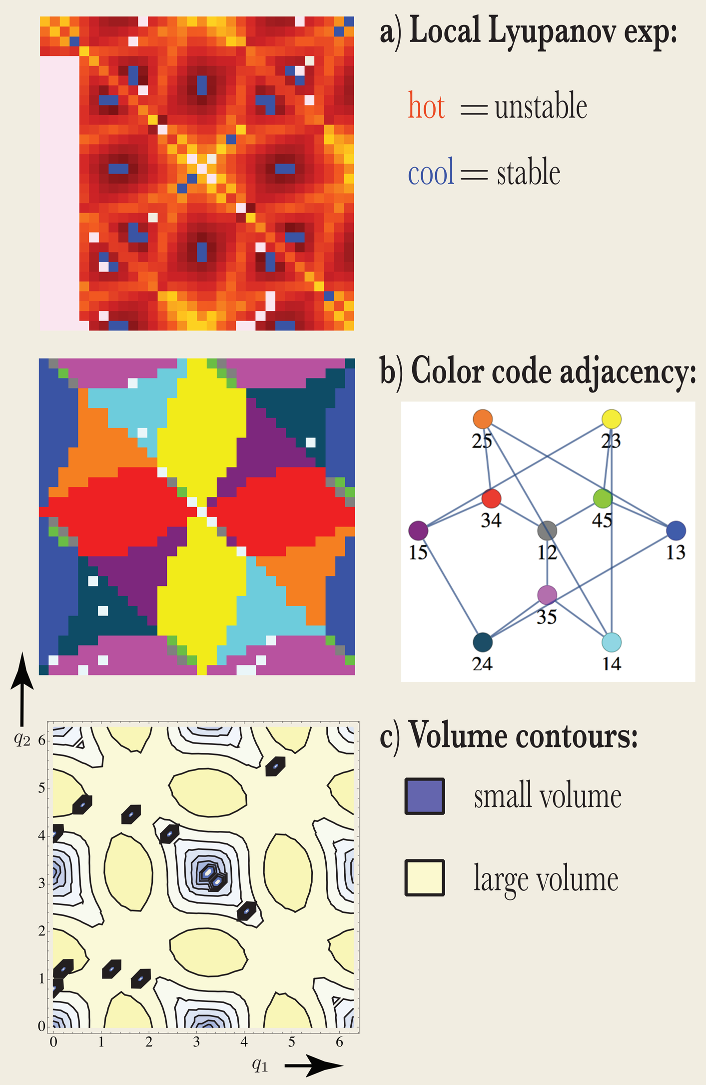
- degenerate levels strongly supressed

By contrast, integrable systems have Poisson level statistics with the probability of small spacing enhanced.



Configuration space, equi-area pentahedron

These data are kindly shared by Coleman-Smith and Müller [3].



Numerical evidence from [1] and [3] suggests that the classical volume evolution generated by the Hamiltonian $H = V_{\text{pent}}$ is chaotic.

For example, the local Lyupanov exponents of panel **a**) clearly show that the boundaries between adjacency regions [see **b**)] are unstable.

Panel **c**) illustrates that the contours of constant volume, and hence the volume evolution, frequently cross over adjacency boundaries. Note that the smallest physical volumes occupy a small region of the phase space.

The level repulsion of chaotic quantum systems together with the small phase space available at low volumes, i.e. low density of states at small volume, yields our main conclusion:

a volume gap is robust in loop gravity.

Conclusions

These results uncover a new mechanism for the presence of a volume gap in the spectrum of quantum gravity: the level repulsion of quantum systems corresponding to classically chaotic dynamics. The generic presence of a volume gap further strengthens the expected ultraviolet finiteness of quantum gravity theories built on spectral discreteness. We find:

- Robust volume gap due to: chaos & low density of states at low volume
- Pentahedral volume dynamics implies quantum volume states are spread over adjacencies
- Loop gravity continues to indicate physical cutoffs at the Planck scale

[1] H.M. Haggard, PRD **87**, (044020) 2013, [2] E. Bianchi and H.M. Haggard, PRL **107**, (011301) 2011
[3] C. Coleman-Smith and B. Müller, PRD **87**, (044047) 2013

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