3d gravity and quantum groups - Day 2

Bard Summer School on Quantum Gravity

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I. SOME ADDITIONAL DEFINITIONS, THEOREM USEFUL FOR TODAY LECTURE

Theorem Let G be a Lie group with Lie algebra g. If G is a Poisson Lie group, then $\mathfrak{g}$ has a natural Lie bialgebra structure, called the tangent Lie bialgebra of G.
Conversely, if G is connected and simply-connected, every Lie bialgebra structure on $\mathfrak{g}$ is the tangent Lie bialgebra of a unique Poisson structure on G, which makes G into a Poisson Lie group.

Definition A pair $(\mathfrak{g}, \delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g})$ is a *Lie bialgebra* if $\mathfrak{g}$ is a Lie algebra and δ satisfies

- δ is a Lie cobracket which means that $\delta^* = \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ is a Lie bracket.
- a compatibility condition:

$$\delta([x, y]) = [x, \delta(y)] + [\delta(x), y]. \tag{1}$$

  This is a cocycle property of δ. The bracket on $\mathfrak{g}$ can be extended to wedge powers of $\mathfrak{g}$ by declaring on pure tensors that $[x, y \wedge z] \equiv [x, y] \wedge z + y \wedge [x, z]$.

Proposition If $(\mathfrak{g}, \delta)$ is a Lie bialgebra and μ is the Lie bracket of $\mathfrak{g}$, then $(\mathfrak{g}^*, \mu^*)$ is a Lie bialgebra, where δ^* is the Lie bracket of $\mathfrak{g}^*$.

II. PROBLEM SOLVING SESSION

A. Poisson Manifolds

A Poisson manifold is a smooth manifold associated with a Poisson bracket $\{\cdot, \cdot\}$.

$$\{\cdot, \cdot\} : \begin{cases} C^\infty(M) \times C^\infty(M) & \to & C^\infty(M) \\ (f, h) & \to & \{f, h\} \end{cases} \tag{2}$$

The Poisson bracket defines a bivector π

$$\{f, h\}(x) = \langle \pi_x, df \otimes dh \rangle = \pi_x(df, dh), \tag{3}$$

with $\pi_x \in \Lambda^2 T_x M$ and $df, dh \in T_x^* M$ where T_M^* the cotangent space.

1. *Bivectors*

- If π is a bivector, at each point x, π_x has skew-symmetric components in local coordinates $(\pi_x)_{ij}$, $i, j = 1, 2, \cdots, \dim M$.
- At each point $x \in M$, we can view π_x as a skew-symmetric bilinear form on $T_x^* M$, or as the the skew-symmetric linear map $\underline{\pi}_x$ from $T_x^* M$ to $T_x M$ such that

$$\langle \eta_x, \underline{\pi}_x(\xi_x) \rangle = \pi_x(\xi_x, \eta_x) \tag{4}$$

  for $\xi_x, \eta_x \in T_x^* M$.

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- If ξ, η are differential 1-form on M, we can define $\pi(\xi,\eta)$ to be the function in $C^\infty(M)$ whose value at $x \in M$ is $\pi_x(\xi_x,\eta_x)$.
- If f and g are functions on M, and df, dg denote their differentials, we set

$$\{f,g\} = \pi(df,dg). \tag{5}$$

- Note that $\underline{\pi}(df)$ is a vector field denoted by X_f, and that

$$\{f,g\} = X_f g \tag{6}$$

So equivalently, a Poisson manifold can be defined as a manifold M with a Poisson bivector π such that (5) satisfies the Jacobi identity. The Jacobi identity translates into an equation written in terms of local coordinates of the bivector.

$$(\pi_x)_{ri}(\pi_x)_{jk,r} + (\pi_x)_{rj}(\pi_x)_{ki,r} + (\pi_x)_{rk}(\pi_x)_{ij,r} = 0 \tag{7}$$

This condition is indeed necessary and sufficient for a bivector to be a Poisson bivector. Then, when (M,π) is a Poisson manifold, $\{f,g\}$ is called the Poisson bracket of f and $g \in C^\infty(M)$ and $X_f = \underline{\pi}(df)$ is called the Hamiltonian vector field with Hamiltonian f.

1. If $M = \mathbb{R}^{2n}$ with coordinates q^i, p_i, $i = 1, \cdots, n$ and if

$$\underline{\pi}(dq^i) = -\frac{\partial}{\partial p_i}, \quad \underline{\pi}(dp_i) = \frac{\partial}{\partial q^i},$$

   write explicitly X_f, $\{f,g\}$ and π for f, $g \in \mathbb{C}^\infty(\mathbb{R}^{2n})$.

2. Action of a symmetry group over a Poisson manifold

Let's $(M,\{\cdot,\cdot\})$ be a Poisson manifold. We are interested in the action of a symmetry group that is consistent with $\{\cdot,\cdot\}$, that is, that this action is a Poisson map. Let's consider as an example the manifold $M = \mathbb{C}^2$ with the canonical Poisson brackets $\{z_i,\, \bar{z}_j\}_M = -i\delta_{ij}$ with $i,j \in \{1,2\}$. We consider the action of $G = \mathrm{SU}(2)$. Let $g \in \mathrm{SU}(2)$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Under SU(2) transformation, the coordinates transform as

$$z_i \;\to\; z_i' = g_{ij} z_j, \qquad \bar{z}_i \;\to\; \bar{z}_i' = \bar{g}_{ij}\bar{z}_j.$$

We want that this action, $\triangleright$, to be consistent with the Poisson brackets.

$$g \triangleright \{f,g\}(x) = \{f,h\}(g \triangleright x) = \{f,h\}_G(g \triangleright x) + \{f,h\}_M(g \triangleright x).$$

1. Show that in this case, i.e. when $\{z_i,\, \bar{z}_j\}_M = -i\delta_{ij}$, the Poisson brackets on G are trivial.
2. Compute $\{g_{ij}, \bar{g}_{kl}\}_G$ if now the Poisson structure for the z's is given by

$$\{z_1,z_2\} = \frac{i}{\beta} z_1 z_2, \quad \{z_1,\bar{z}_2\} = \frac{i}{\beta} z_1\bar{z}_2, \quad \{z_1,\bar{z}_1\} = -i(1 - \frac{2}{\beta} z_1\bar{z}_1), \quad \{z_2,\bar{z}_2\} = -i(1 - \frac{2}{\beta}\sum_{k_1}^{2} |z_k|^2).$$

3. Is this Poisson structure on the group multiplicative (that is that the group multiplication and the Poisson bracket on the group are compatible)?