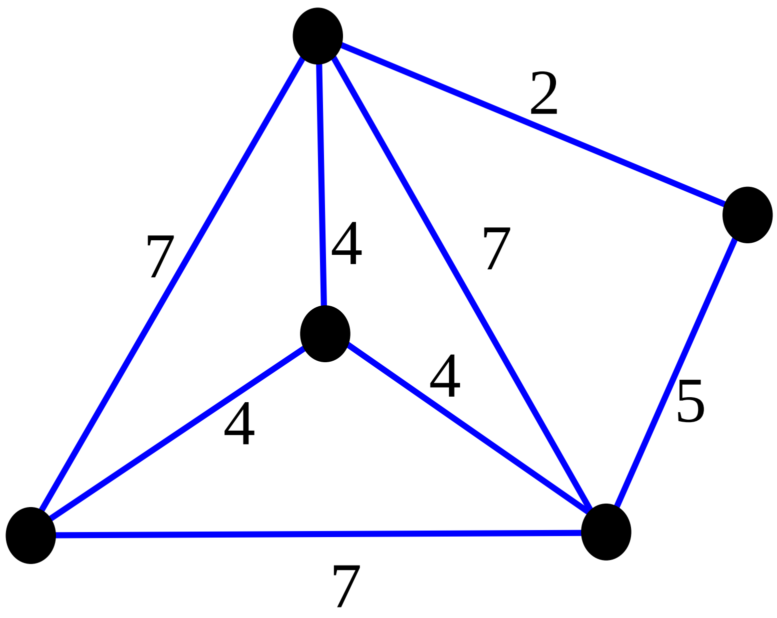
Realizability of Graphs

Discrete Math Day 2008

Maria Belk and Robert Connelly

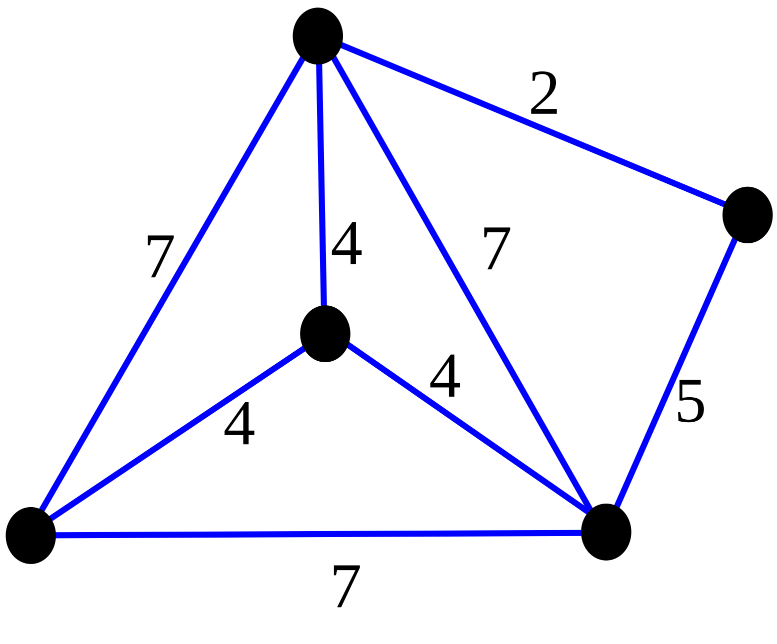
Motivation: the molecule problem

A chemist determines the distances between atoms in a molecule:



Motivation: the molecule problem

1. Is this a possible configuration of points in 3 dimensions?
2. What is a possible configuration satisfying these distances?



Motivation: the molecule problem

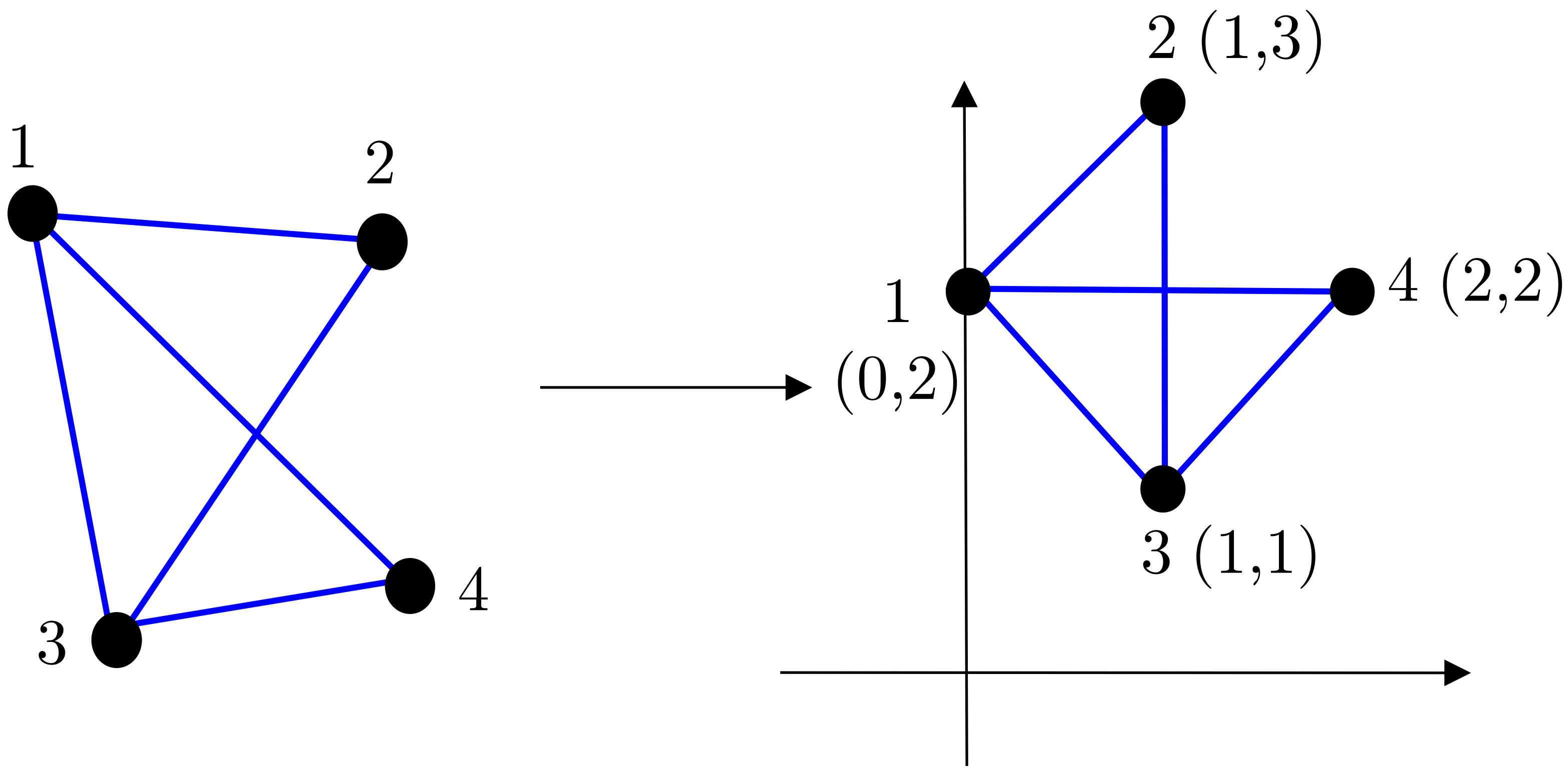
1. Is this a possible configuration of points in 3 dimensions?
2. What is a possible configuration satisfying these distances?

Unfortunately, these questions are NP-hard.

We will answer a different question.

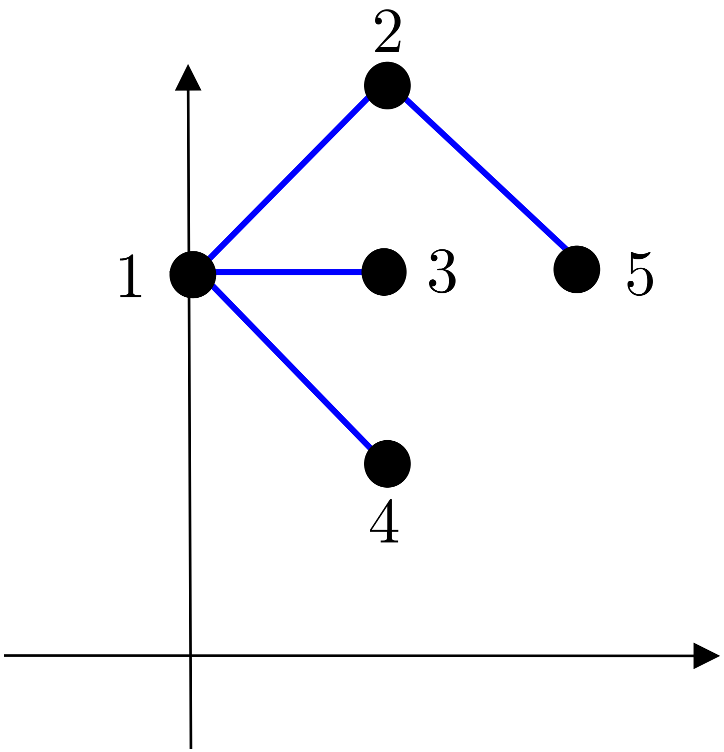
Realization

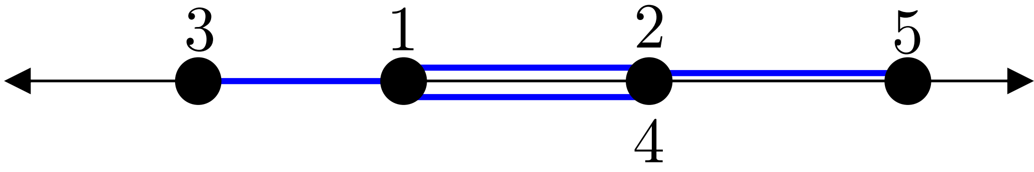
Realization: A **realization** of a graph G is a placement of the vertices in some $\mathbb{R}^d$.



Realization

Here are two realizations of the same graph:

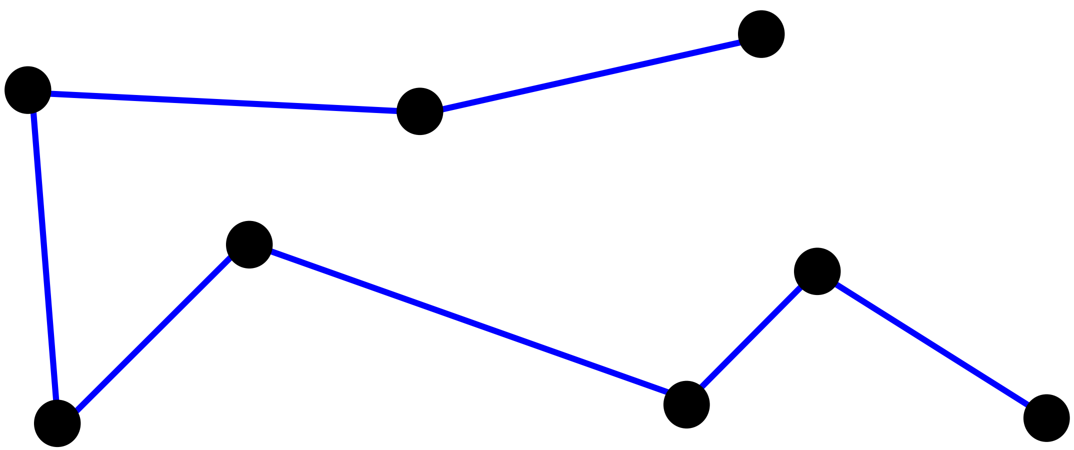




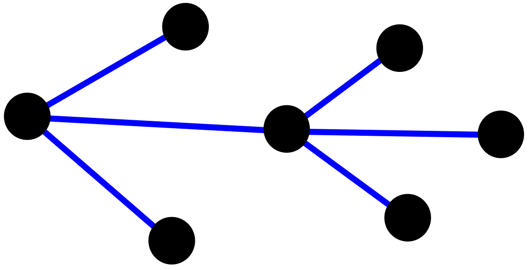
d-realizability

$\boldsymbol{d}$-realizable: A graph is **$\boldsymbol{d}$-realizable** if given any realization of the graph in some $\mathbb{R}^n$ (possibly high dimensional), there exists a realization in $\mathbb{R}^d$ with the same edge lengths.

Example: A path is 1-realizable.



Examples



- A tree is 1-realizable.

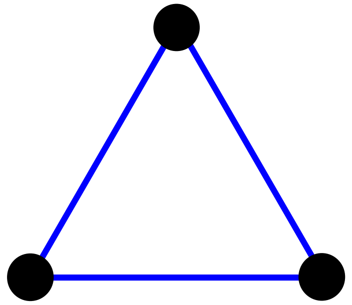
- K_3 is 2-realizable, but not 1-realizable.

- A cycle is also 2-realizable, but not 1-realizable.

- Any graph containing a cycle is not 1-realizable.

Examples

- A tree is 1-realizable.
- K_3 is 2-realizable, but not 1-realizable.



- A cycle is also 2-realizable, but not 1-realizable.
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Examples

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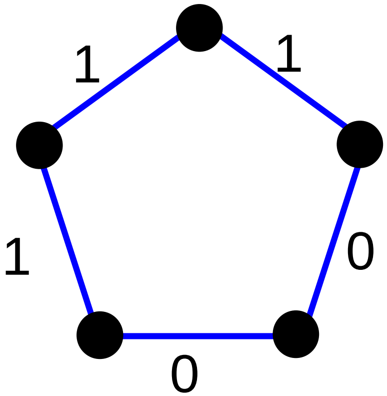
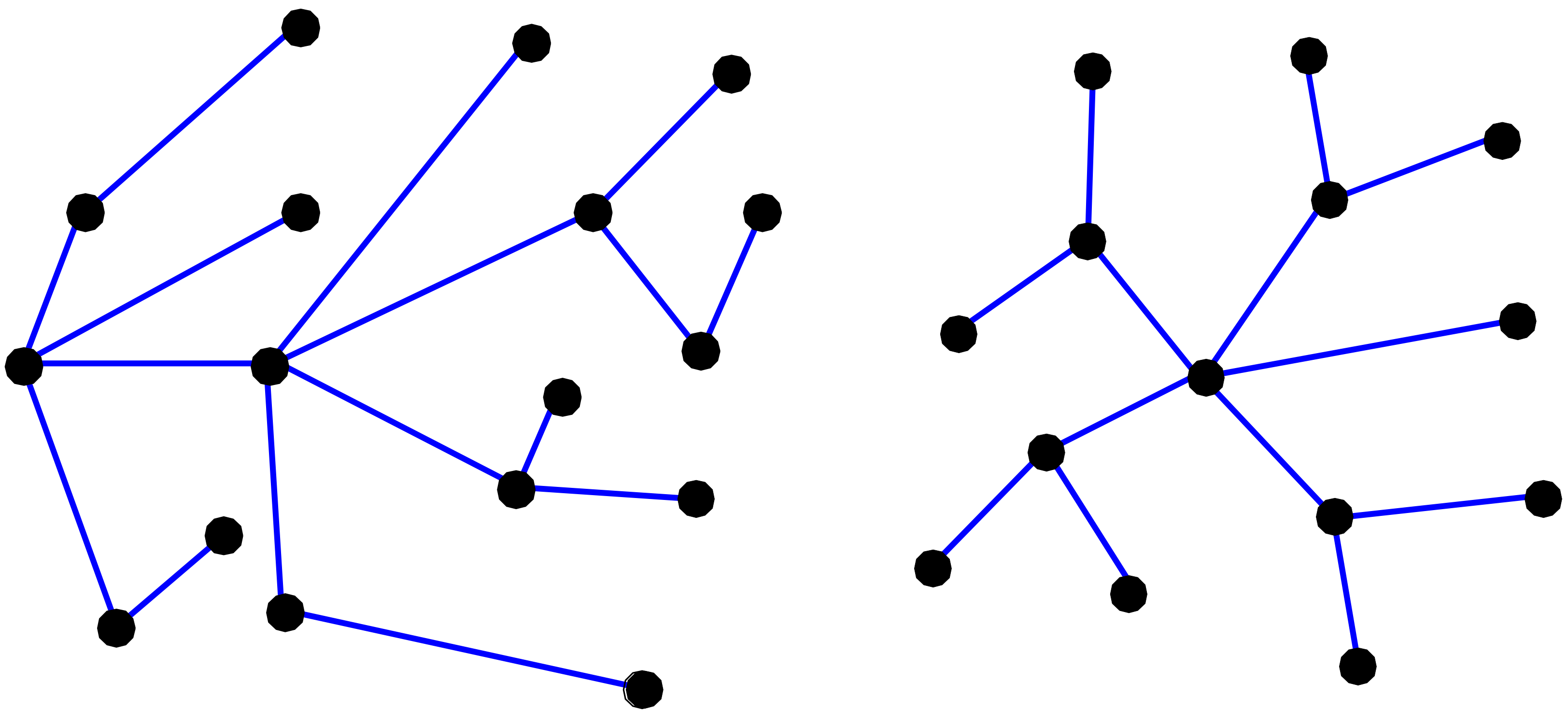


- Any graph containing a cycle is not 1-realizable.

Examples

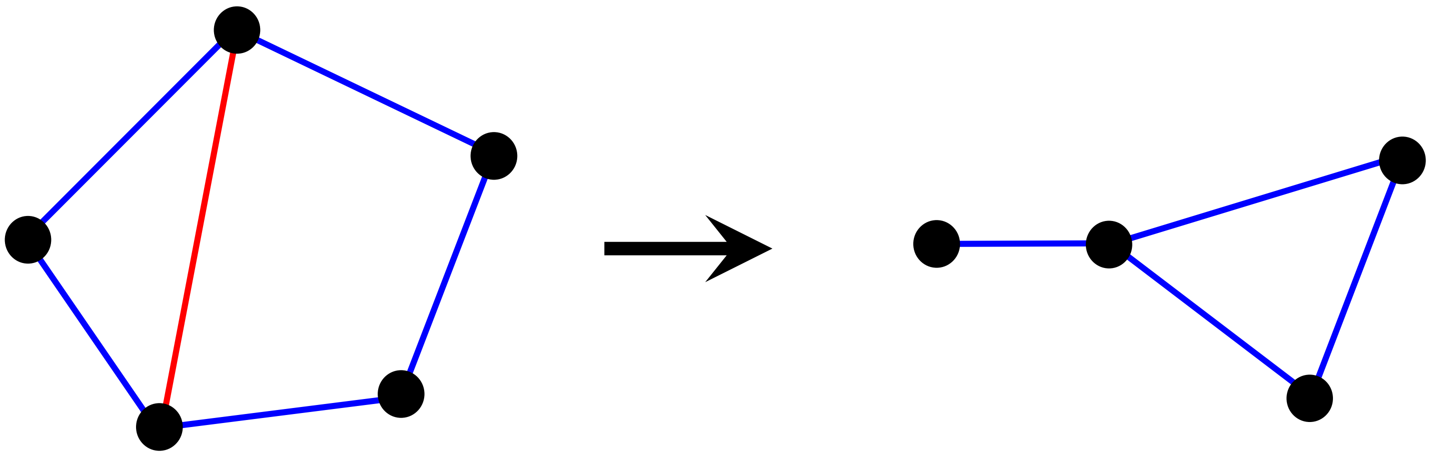
- A tree is 1-realizable.
- K_3 is 2-realizable, but not 1-realizable.
- A cycle is also 2-realizable, but not
1-realizable.
- Any graph containing a cycle is not 1-realizable.

Theorem. (Connelly) A graph is 1-realizable if and only if it a forest (a disjoint collection of trees).



Definition. A **minor** of a graph G is a graph obtained by a sequence of

- Edge deletions and
- Edge contractions (identify the two vertices belonging to the edge and remove any loops or multiple edges).



Definition. A **minor** of a graph G is a graph obtained by a sequence of

- Edge deletions and
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Theorem. (Connelly) If G is d-realizable then every minor of G is d-realizable (this means d-realizability is a minor monotone graph property).

Graph Minor Theorem

Theorem (Robertson and Seymour). For a minor monotone graph property, there exists a finite list of graphs G_1, …, G_n such that a graph G satisfies the minor monotone graph property if and only if G does not have G_i as a minor.

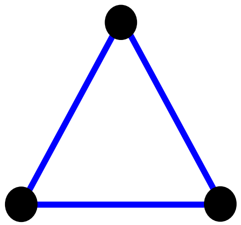
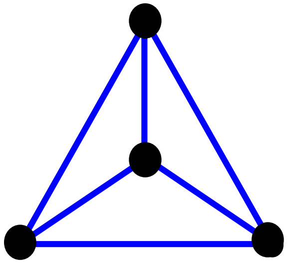
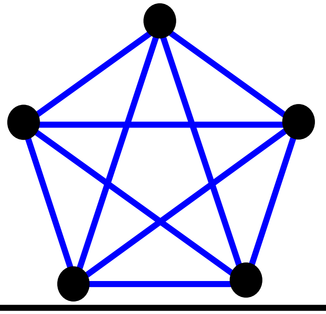
Example: A graph is 1-realizable if and only if it does not contain K_3 as a minor.

Graph Minor Theorem

By the Graph Minor Theorem:

For each d, there exists a finite list of graphs G_1, …, G_n such that a graph is d-realizable if and only if it does not have any G_i as a minor.

Forbidden Minors

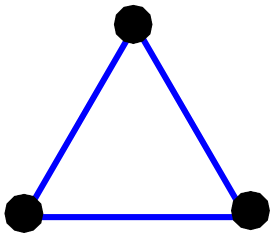
1-realizable	K_3	
2-realizable	K_4+ ?	
3-realizable	K_5 + ?	

Which graphs are 2-realizable?

2-realizability

<u>2-tree</u>:

- Start with a triangle.
- Attach another triangle along an edge.
- Continue attaching triangles to edges.



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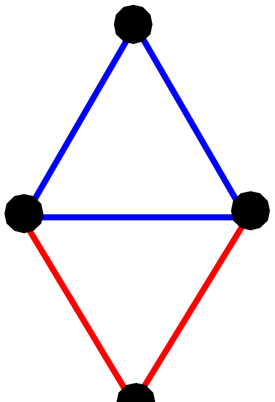

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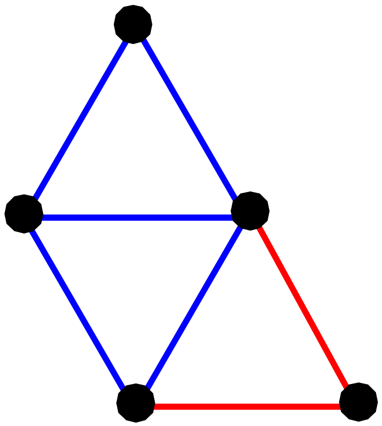

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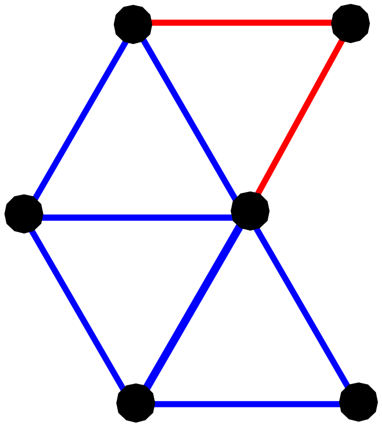

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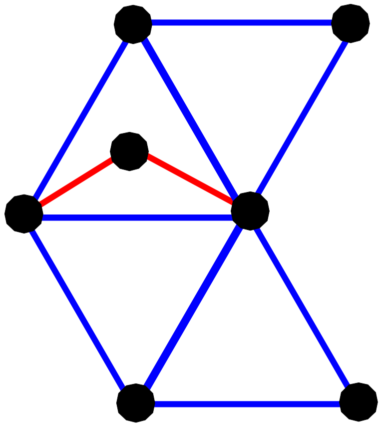

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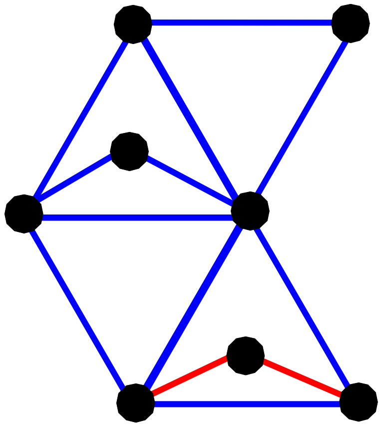

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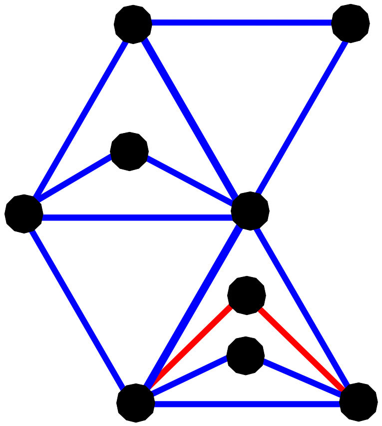

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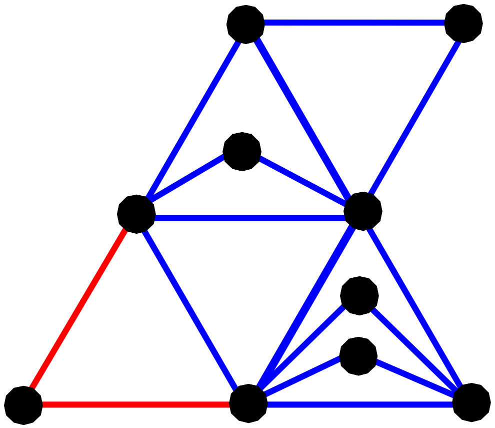

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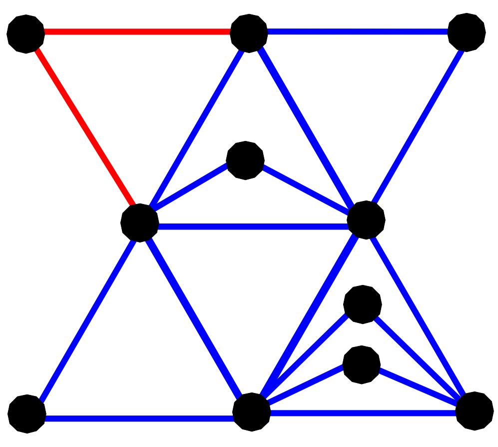

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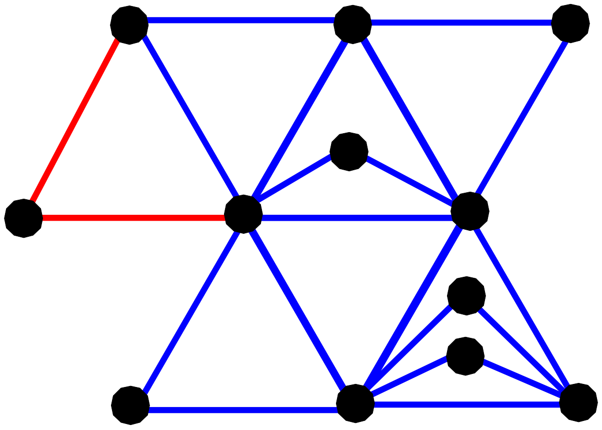

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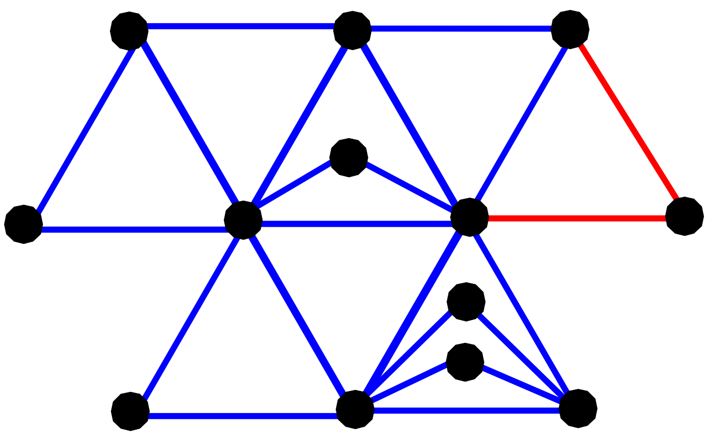

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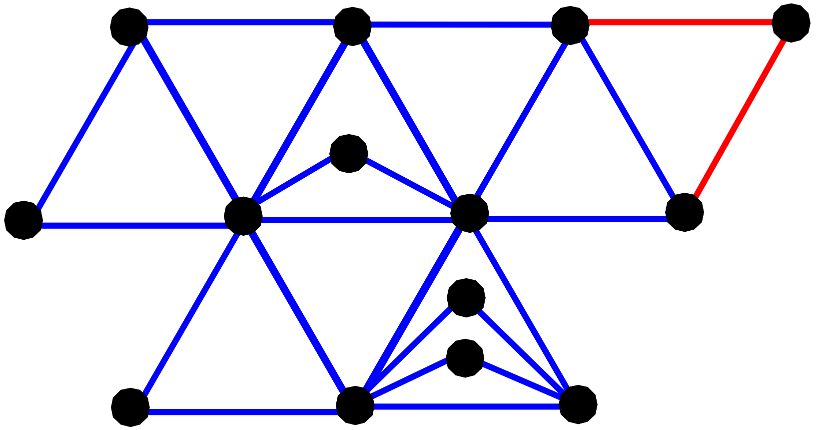
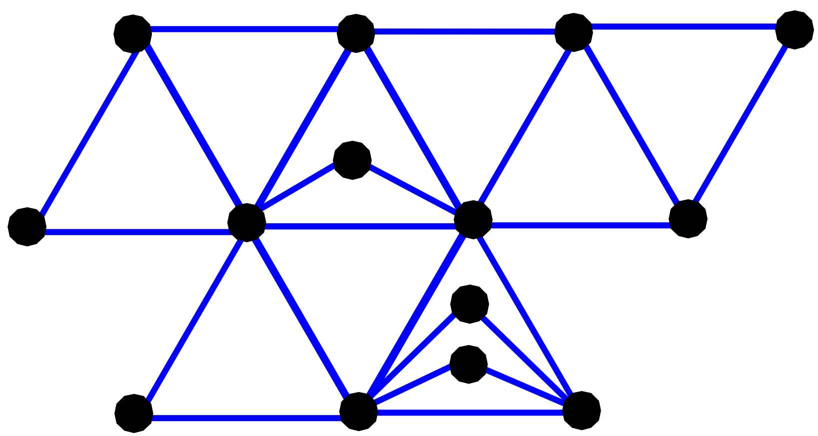


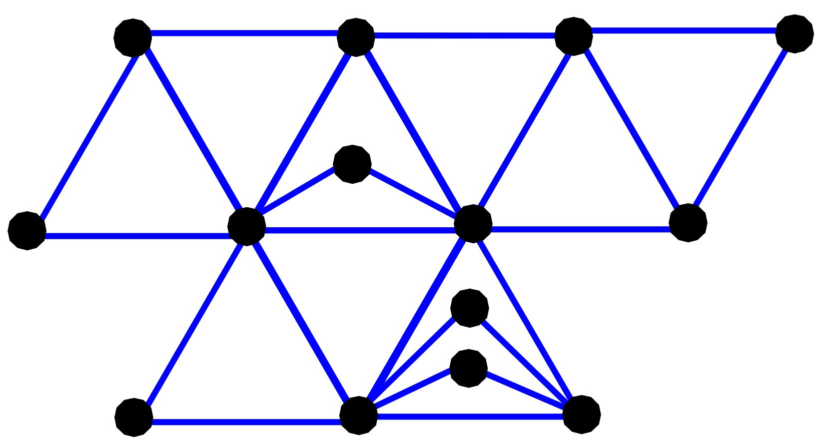
2-realizability

2-trees are 2-realizable.



2-realizability

Partial 2-tree: Subgraph of a 2-tree



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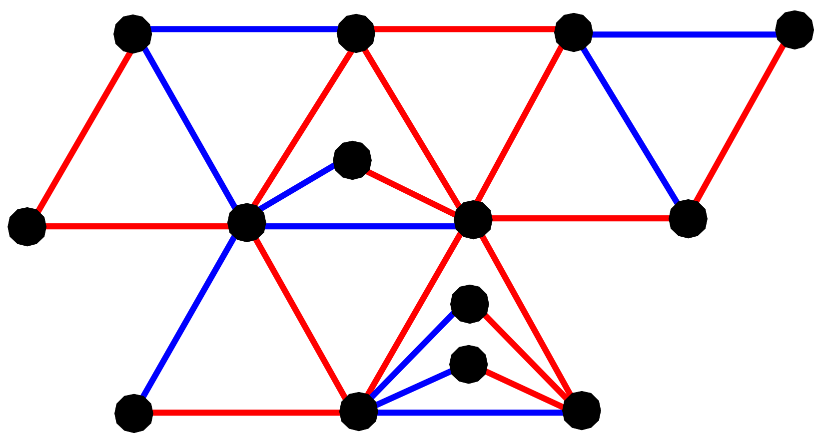

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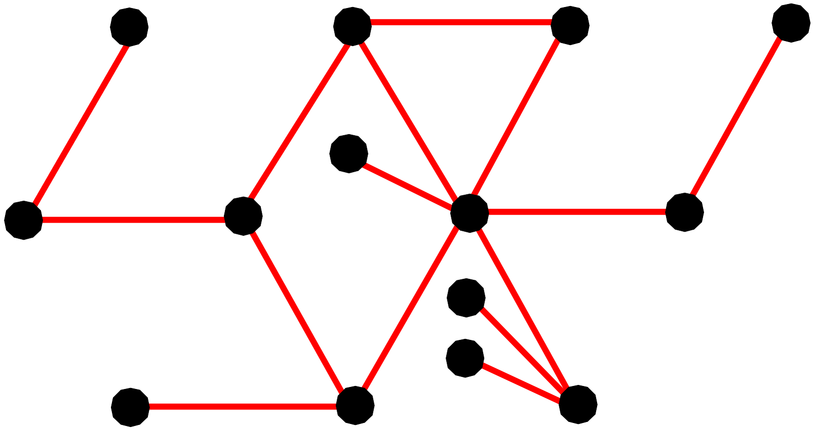
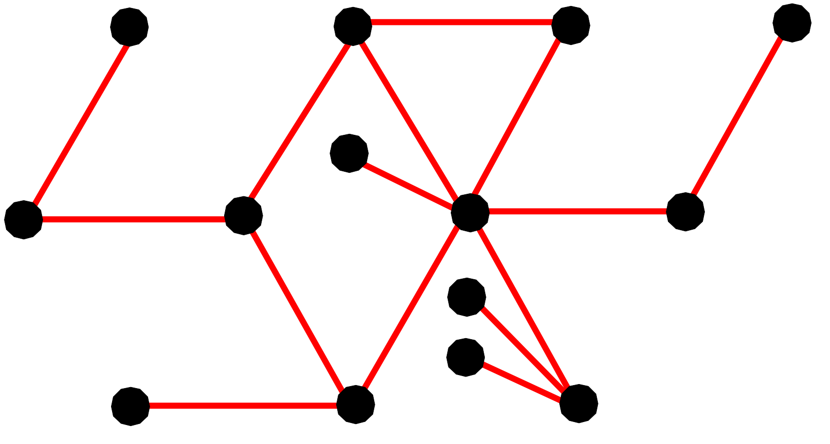


2-realizability

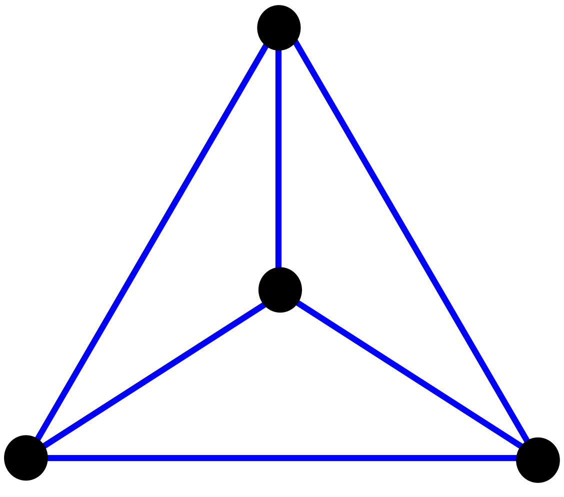
Partial 2-trees are also 2-realizable.



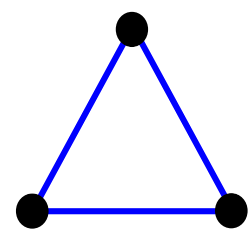
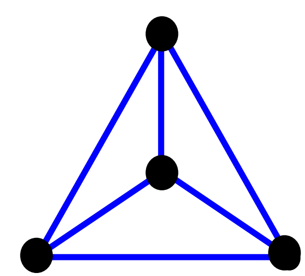
2-realizability

Theorem (Wagner, 1937). G is a partial *2*-tree if and only if G does not contain K_4 as a minor.

Theorem (Belk, Connelly). G is *2*-realizable if and only if G does not contain K_4 as a minor.



Realizability

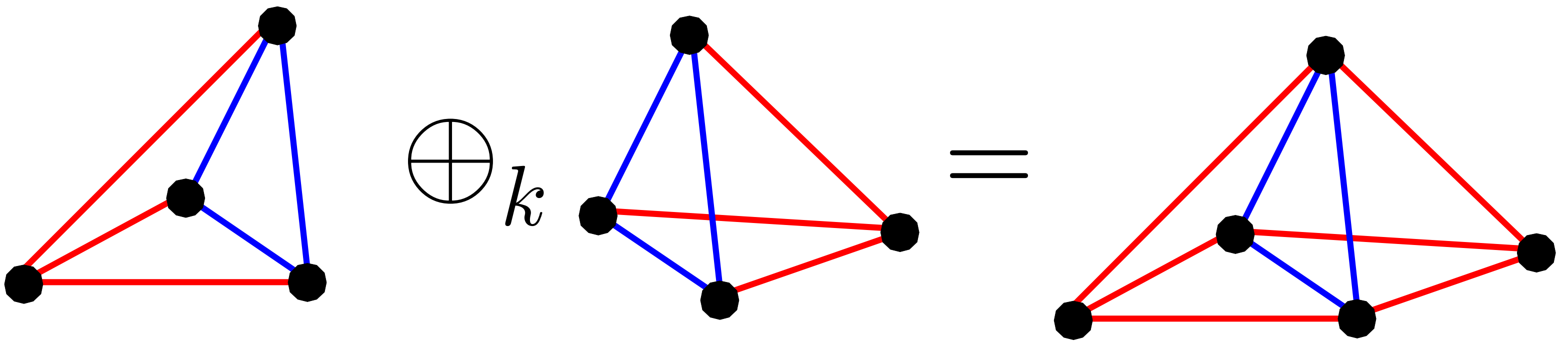
	Allowed	Forbidden
1-realizability	Trees	
2-realizability	Partial 2-trees	

Which graphs are 3-realizable?

3-realizability

***k*-sum**:

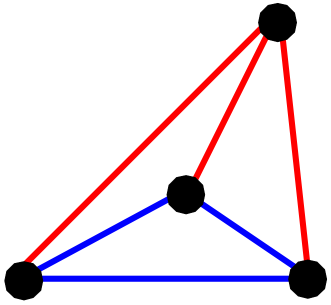
- G_1 contains a K_k subgraph
- G_2 contains a K_k subgraph
- $G_1 \oplus_k G_2$ is obtained by identifying the K_k subgraphs



3-realizability

k-tree:

- Start with a K_{k+1}.
- k-sum with another K_{k+1}.
- Continue k-summing with K_{k+1}.



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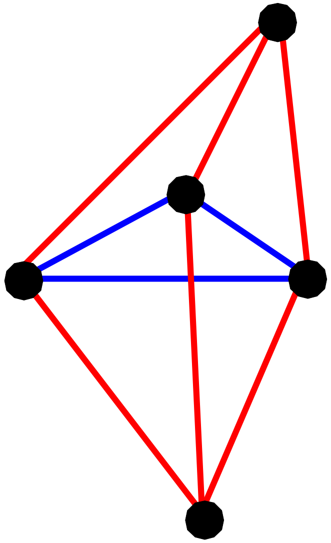

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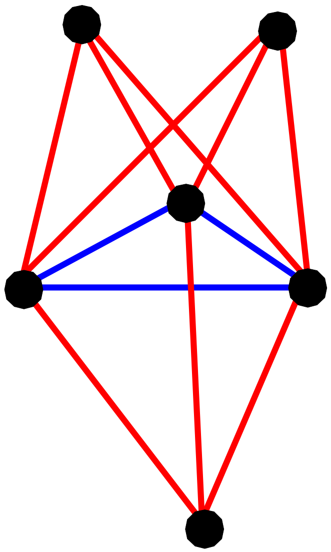
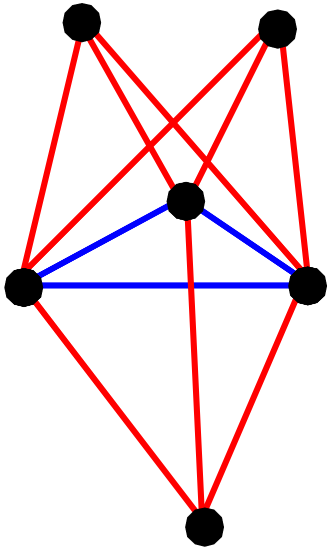


3-realizability

Partial k-tree: Subgraph of a k-tree



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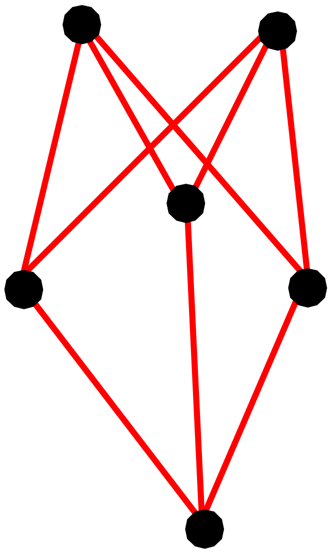
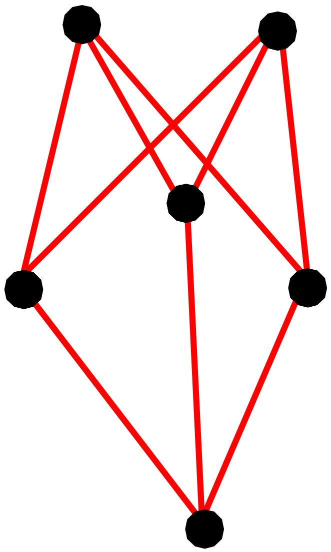


3-realizability

Theorem (Connelly) All partial d-trees are d-realizable.



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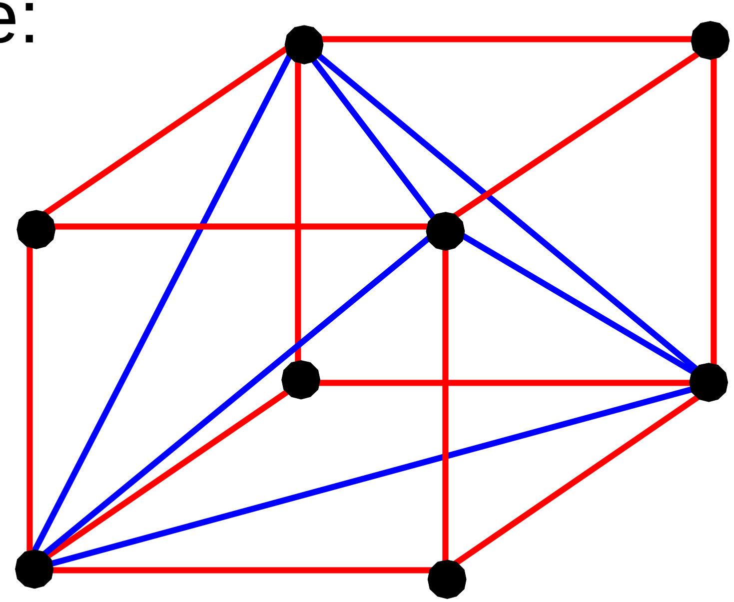
Another example:



3-realizability

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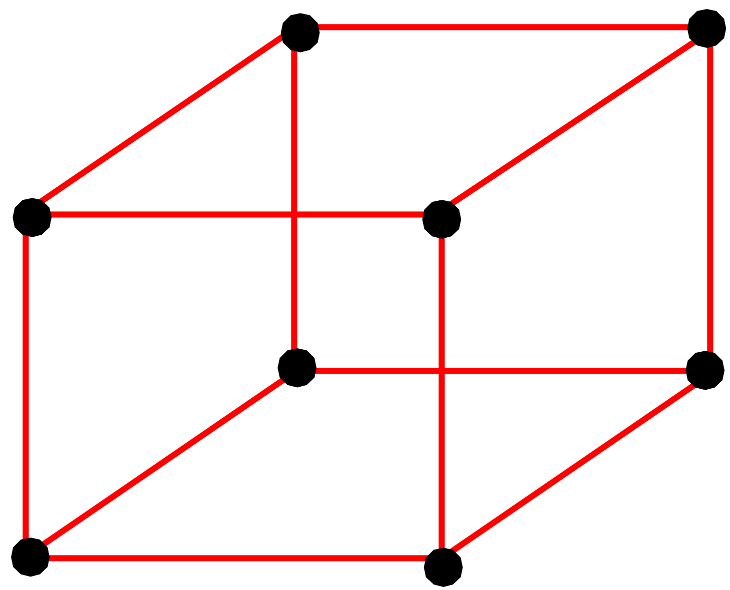


3-realizability

Are the following all equal?

- Partial 3-trees
- Not containing K_5
- 3-realizability

3-realizability

Are the following all equal?

- Partial 3-trees
- Not containing K_5
- 3-realizability

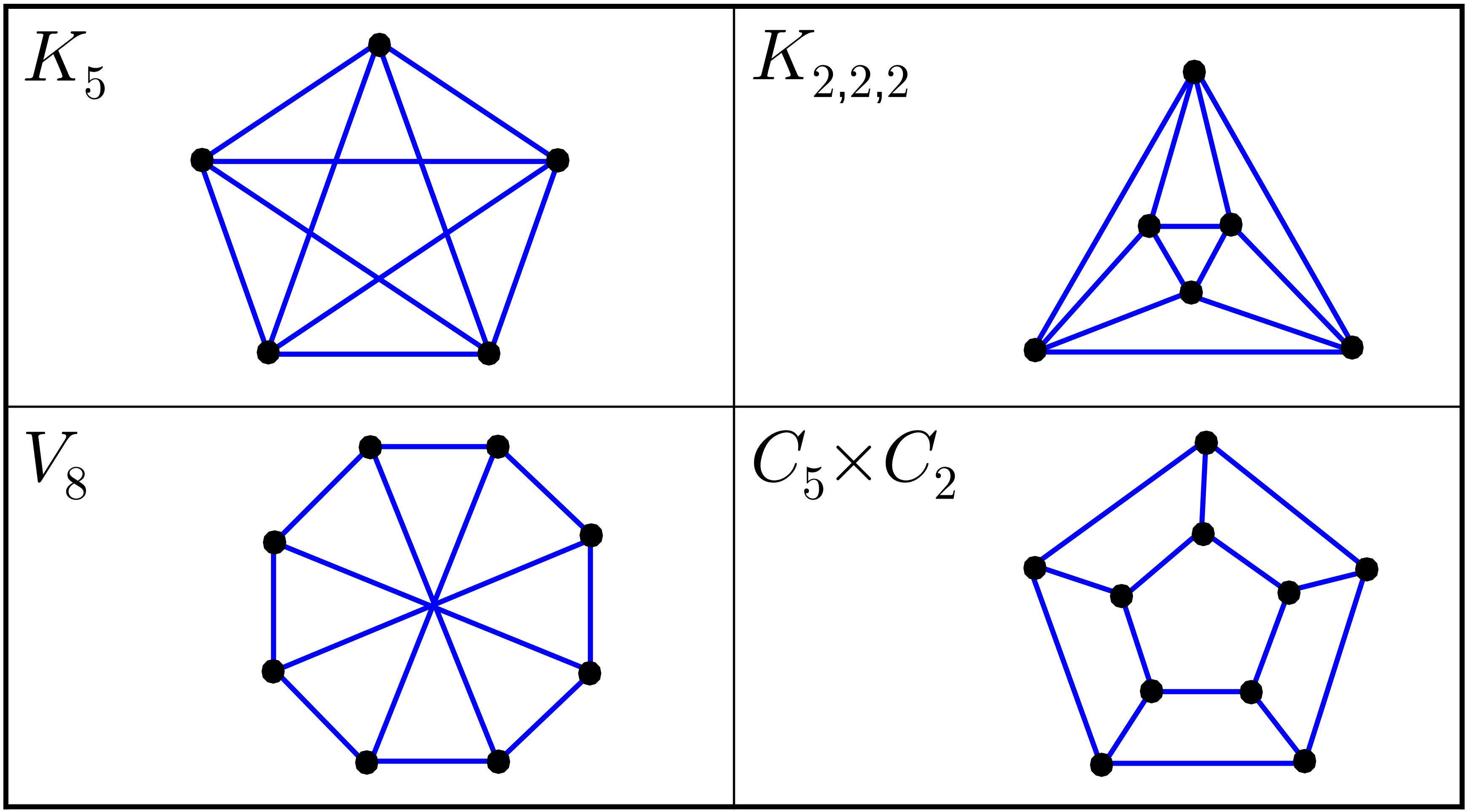
Answer: No, none of the three are equal.

3-realizability

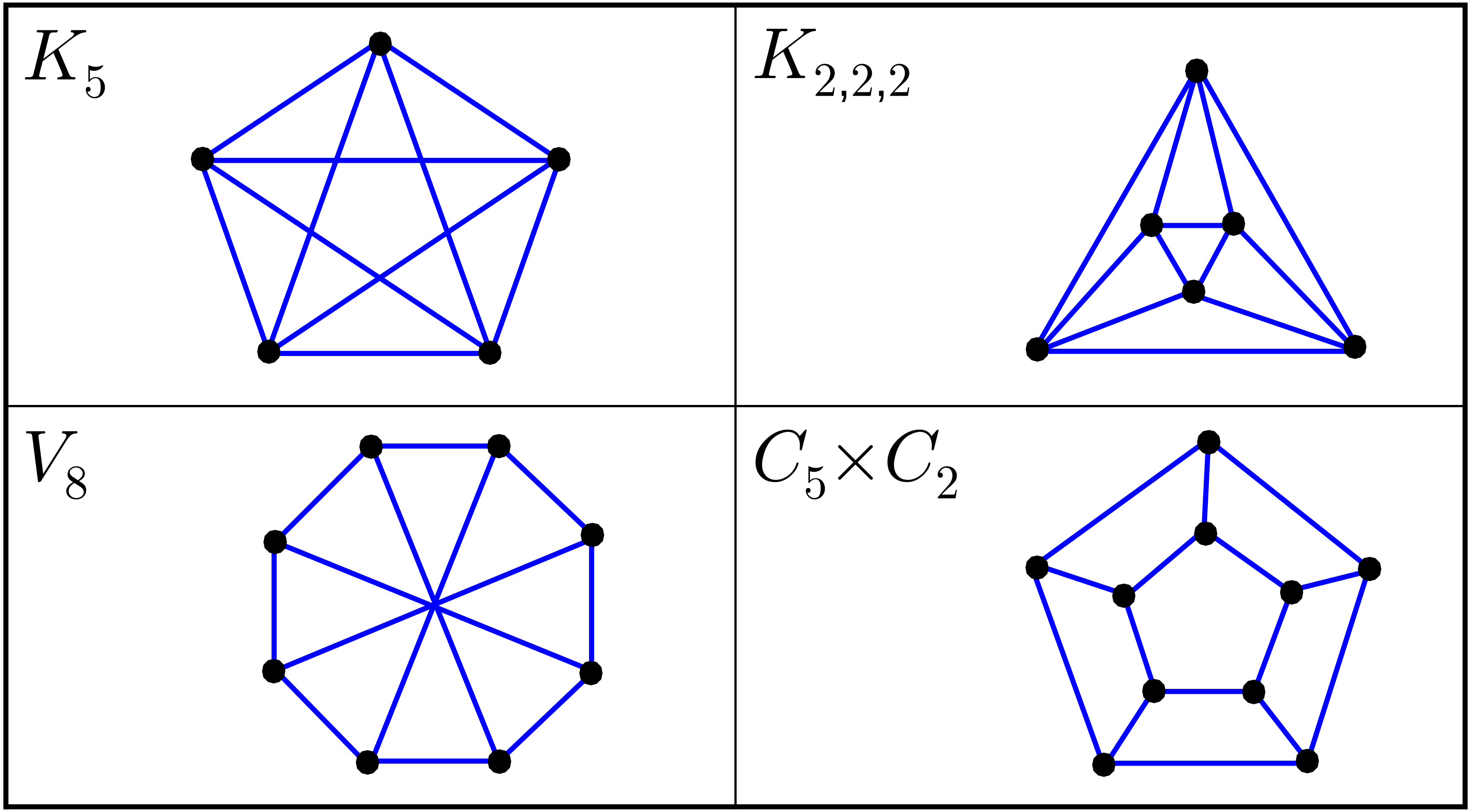
Partial 3-trees	$\Rightarrow$	3-realizability	$\Rightarrow$	Does not contain K_5

None of the reverse directions are true.

Theorem (Arnborg, Proskurowski, Corneil)
The forbidden minors for partial 3-trees.



Which of these graphs are 3-realizable?



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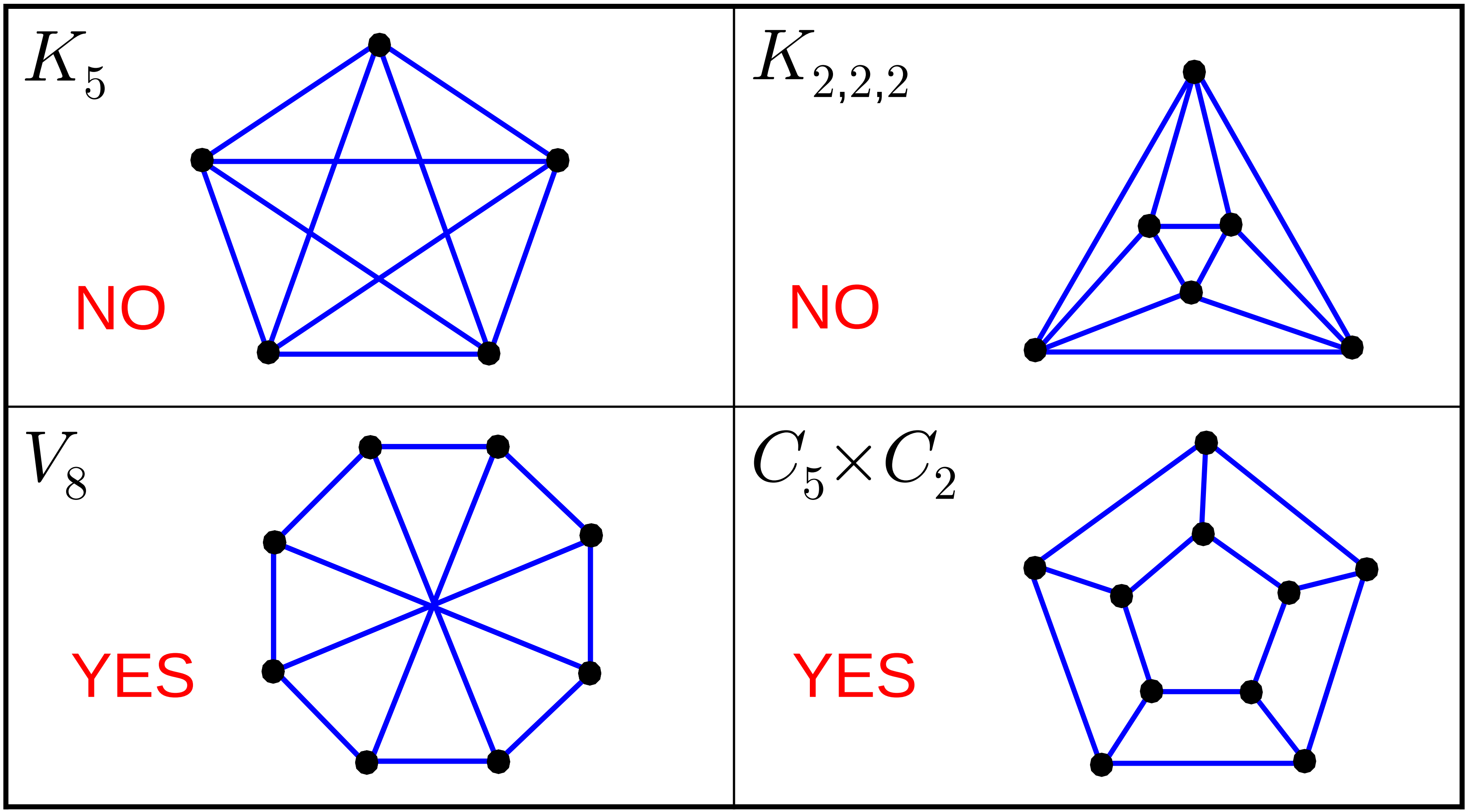

Are there any more forbidden minors for 3-realizability?

Lemma (Belk and Connelly). If G contains V_8 or $C_5 \times C_2$ as a minor then either

- G contains K_5 or $K_{2,2,2}$ as a minor, or
- G can be constructed by 2-sums and 1-sums of partial 3-trees, V_8's, and $C_5 \times C_2$'s (and is thus 3-realizable).

Theorem (Belk and Connelly). The forbidden minors for 3-realizability are K_5 and $K_{2,2,2}$.

Are there any more forbidden minors for 3-realizability? NO

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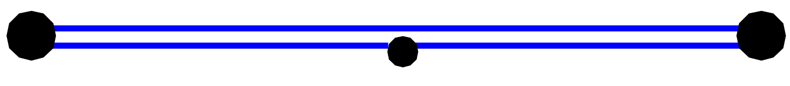
$K_{2,2,2}$ is not 3-realizable



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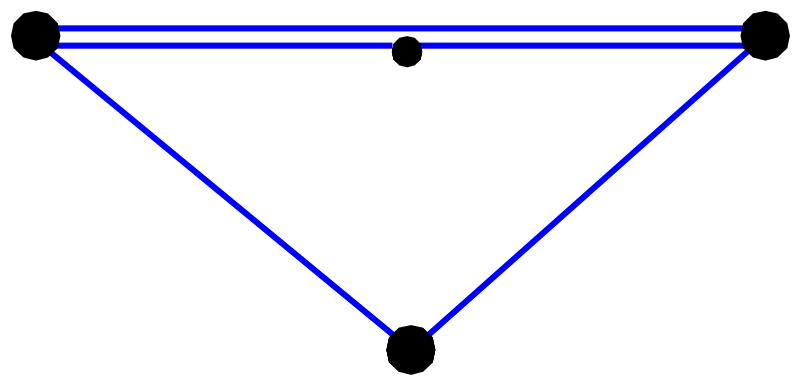

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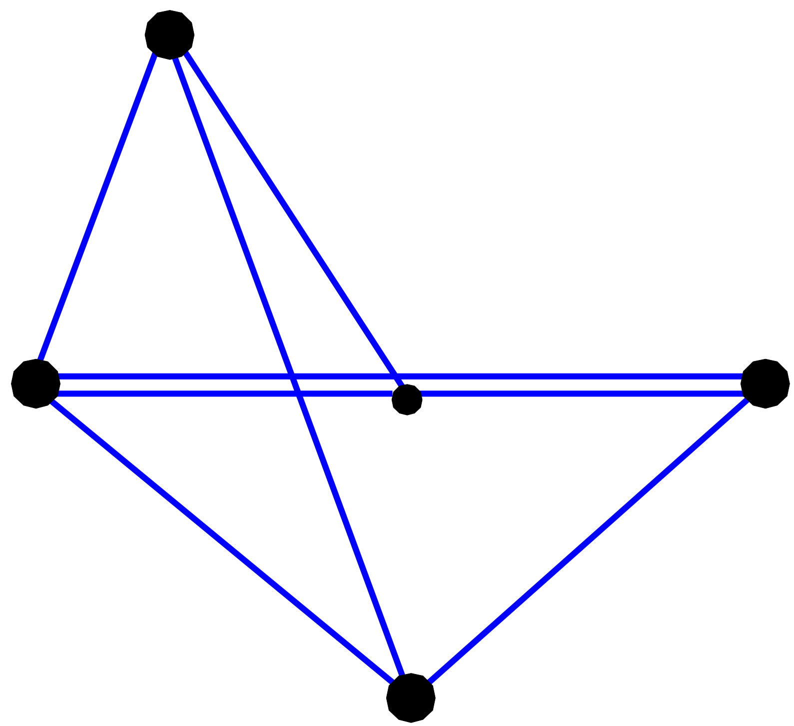

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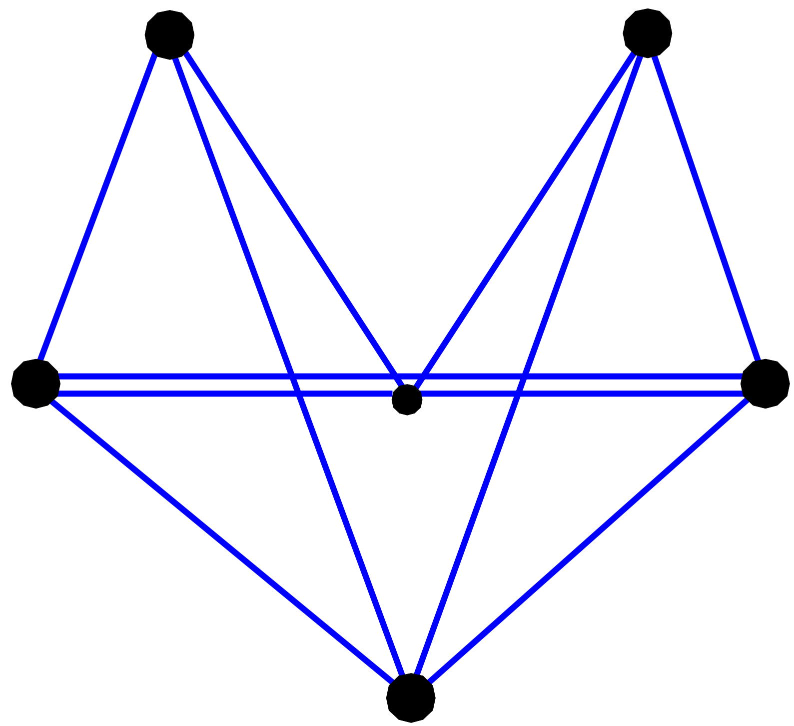

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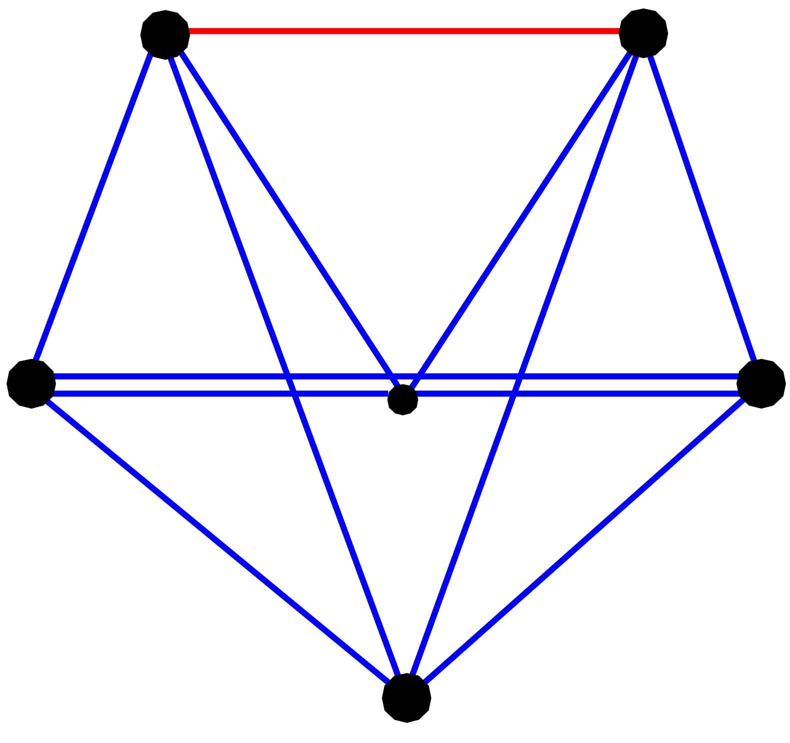

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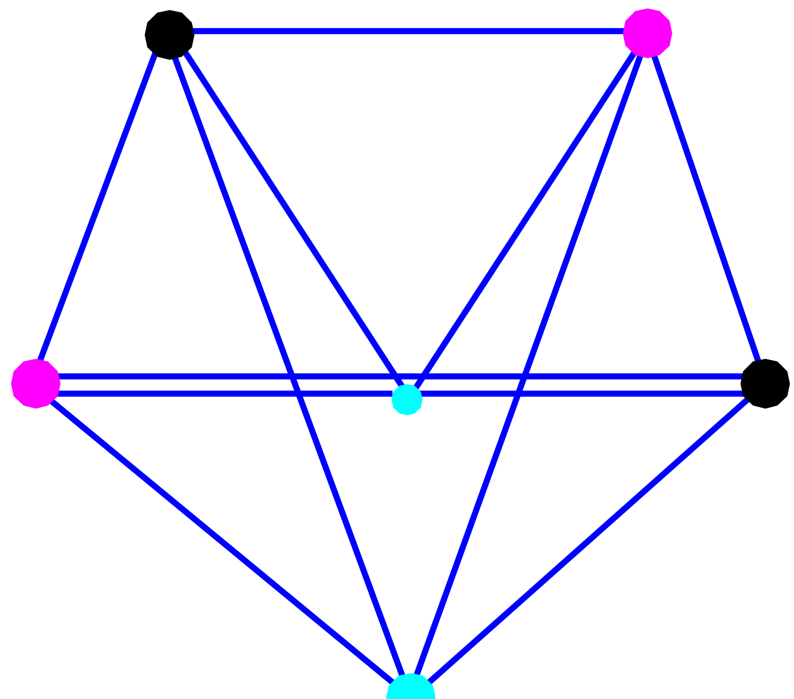
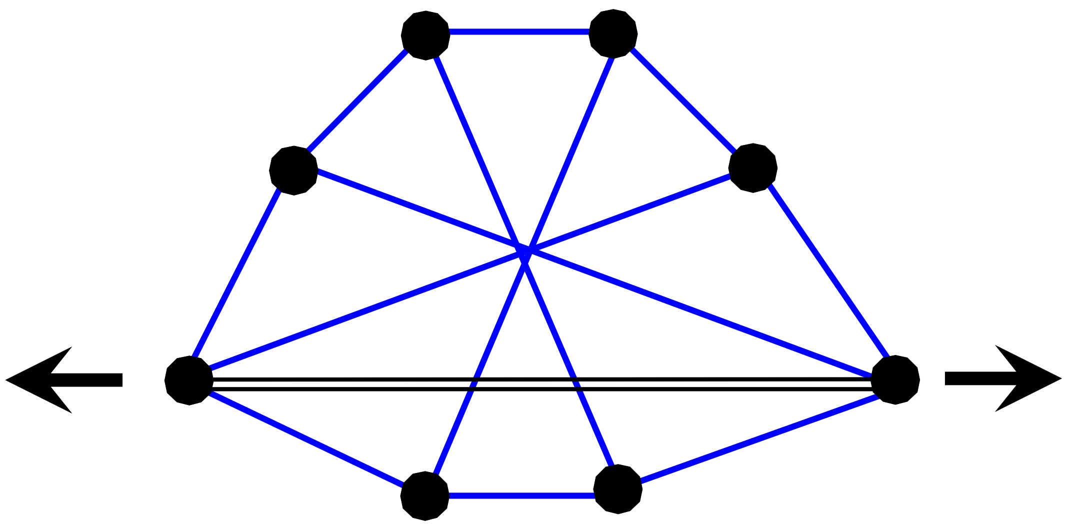


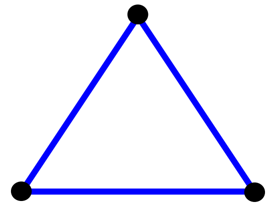
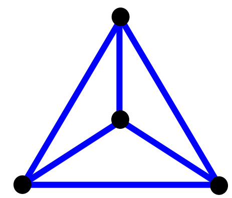
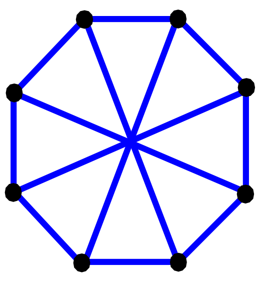
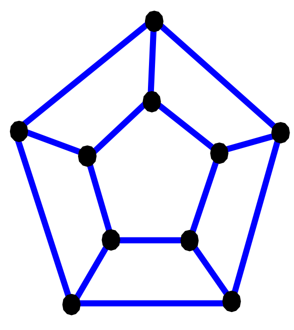
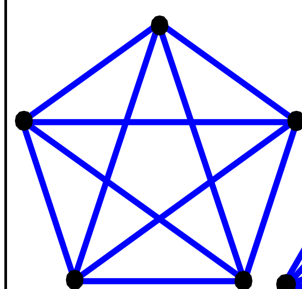
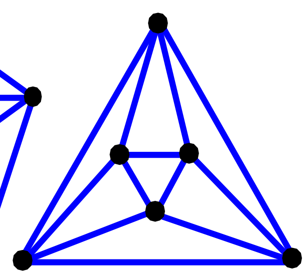
V_8 and $C_5 \times C_2$

Theorem (Belk). V_8 and $C_5 \times C_2$ are 3-realizable.

Idea of Proof: Pull 2 vertices as far apart as possible.



Conclusion

	Allowed	Forbidden
1-realizable	Trees	
2-realizable	Partial 2-trees	
3-realizable	Partial 3-trees  	 

4-realizability?

Which graphs are 4-realizable?

- K_6 is a forbidden minor.
- There is an obvious generalization of $K_{2,2,2}$, which is a forbidden minor.
- However, there are over 75 forbidden minors for partial 4-trees.