MATH 352: Written Assignment 2 **Due: Friday, February 14, 2020**

Show all appropriate work.

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1. (a) Let $V = \{x \in \mathbb{R} \mid x > 0\}$ with addition defined by $x + y \equiv xy$ and scalar multiplication defined by $cx \equiv x^c$ for $c \in \mathbb{R}$. Is V a vector space?

   (b) Redefine scalar multiplication in $\mathbb{R}^2$ to be $c < x_1, x_2 > \equiv < cx_1, 0 >$. If we keep the usual definition of addition, is this a vector space?

2. Let $\mathcal{P}_3 = \{\text{Polynomials of } \deg \leq 3\}$.

   (a) Show that $\mathcal{P}_3$ is a vector space.

   (b) What is the dimension of $\mathcal{P}_3$? Justify your answer finding a basis.

   (c) Find a basis for the following subspace of $\mathcal{P}_3$, $S = \{p(x) \in \mathcal{P}_3 |\, p(1) = 0\}$.

3. Recall that for a vector space V, the covector space V^*, the set of all covectors, is the set of all linear maps from V to $\mathbb{R}$. That is, if $\omega \in V^*$ then

$$\omega(a\mathbf{u} + b\mathbf{v}) = a\omega(\mathbf{u}) + b\omega(\mathbf{v}) \in \mathbb{R},$$

   for all $\mathbf{u}, \mathbf{v} \in V$ and $a, b \in \mathbb{R}$.

   (a) Show that V^* is a vector space.

   (b) Justify the following claim: If $\{\hat{\mathbf{e}}_\mathbf{i} \in V |\, i = 1, \ldots, n\}$ is a basis for V then $\{\hat{\theta}^{(i)} \in V^* |\, i = 1, \ldots, n\}$ such that $\hat{\theta}^{(i)}(\hat{\mathbf{e}}_\mathbf{j}) = 1$ if $i = j$ and 0 otherwise, is a basis for V^*.

4. Suppose W and U are subspaces of a vector space V. Define $U + W \equiv \{\mathbf{u} + \mathbf{w} \mid \mathbf{u} \in U,\ \mathbf{w} \in W\}$.

   (a) Show that $U + W$ is a vector space.

   (b) If U and W are both lines through the origin in $\mathbb{R}^n$, what is the difference between $U + W$ and $U \cup W$?

   (c) What is the span of $U \cup W$?

5. Let $\mathbf{q}_1, \ldots, \mathbf{q}_r$ be a basis for $U \cap W$. Extend them with $\mathbf{u}_1, \ldots, \mathbf{u}_s$ to a basis for U. Separately extend the q's with $\mathbf{w}_1, \ldots, \mathbf{w}_t$ to a basis for W.

   (a) Show that the q's, u's and w's together and linearly independent.

   (b) Deduce from part (a) that

$$\dim(U) + \dim(W) = \dim(U \cap W) + \dim(U + W).$$