MATH 352: Written Assignment 4 **Due: Friday, March 6, 2020**

Show all appropriate work.

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1. Read Sean Carroll's lecture notes starting with the paragraph above eqn. (1.44) on pg. 13 through pg. 19 (the last paragraph continues onto pg. 20).

2. Consider a tensor $X^{\mu\nu}$ and vector V^{μ} with components

$$X^{\mu\nu} = \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix}, \quad V^{\mu} = \begin{pmatrix} -1 \\ 2 \\ 0 \\ -2 \end{pmatrix}.$$

Using the Minkowski metric, $\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, find the components of:

(a) $X^{\mu}{}_{\nu}$ $(= \eta_{\nu\lambda} X^{\mu\lambda})$.

(b) $X_{\mu}{}^{\nu}$ $(= \eta_{\mu\lambda} X^{\lambda\nu})$.

(c) $X^{\lambda}{}_{\lambda}$.

(d) $V^{\mu} V_{\mu}$.

(e) $V_{\mu} X^{\mu\nu}$.

(f) $X^{(\mu\nu)} \equiv \frac{1}{2}(X^{\mu\nu} + X^{\nu\mu})$.

(g) $X^{[\mu\nu]} \equiv \frac{1}{2}(X^{\mu\nu} - X^{\nu\mu})$.

3. Let $F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}$ and define the current 4-vector to be $J^{\mu} = (\rho, J^1, J^2, J^3)$ where ρ is the charge density and $\mathbf{J}$ is the current. Read Sean Carroll's lecture notes starting at the first paragraph on pg. 20 through the paragraph following eqn. (1.78). There it is shown that Maxwell's first two equations (the first two lines of eqn. (1.73)) can be written as $\partial_{\mu} F^{\nu\mu} = J^{\nu}$. First show, using the antisymmetry of $F^{\mu\nu}$ $(F^{\mu\nu} = -F^{\nu\mu})$, that $\partial_{[\mu} F_{\nu\lambda]} = 0$ is equivalent to

$$\partial_{\mu} F_{\nu\lambda} + \partial_{\nu} F_{\lambda\mu} + \partial_{\lambda} F_{\mu\nu} = 0.$$

Here $\partial_{[\mu} F_{\nu\lambda]} = \frac{1}{6}(\partial_{\mu} F_{\nu\lambda} - \partial_{\nu} F_{\mu\lambda} + \partial_{\nu} F_{\lambda\mu} - \partial_{\lambda} F_{\nu\mu} + \partial_{\lambda} F_{\mu\nu} - \partial_{\mu} F_{\lambda\nu})$. Finally, show that the last two Maxwell's Equations (last two lines of eqn. (1.73) can be written as $\partial_{[\mu} F_{\nu\lambda]} = 0$. You may need the definition of the Levi-Cevita Symbol, $\epsilon_{\mu\nu\lambda\rho}$ or ϵ_{ijk}, which is given in eqn. (1.57).

4. Consider a boost in the x-direction, $\Lambda^{\mu'}{}_{\mu} = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

(a) Show that the x-axis, $x^{\mu} =< 0, x, 0, 0 >$, and the t-axis, $y^{\mu} =< t, 0, 0, 0 >$, remain orthogonal after performing the boost. I.e. show that $\eta_{\mu'\nu'} x^{\mu'} y^{\nu'} = 0$.

(b) Show that the line $t = x$ is invariant under a boost.