11.1 Bases for Subspaces

We previously defined a basis for a subspace as a minimum set of vectors that spans the subspace. That is, a basis for a \( k \)-dimensional subspace is a set of \( k \) vectors that span the subspace.

Now that we know about linear independence, we can provide a slightly different definition of a basis.

**Basis for a Subspace**

Let \( S \) be a subspace of \( \mathbb{R}^n \). A set \( \{ b_1, b_2, \ldots, b_k \} \) of vectors in \( \mathbb{R}^n \) is called a **basis** for \( S \) if the following conditions are satisfied:

1. \( \text{Span}\{b_1, b_2, \ldots, b_k\} = S \).
2. The vectors \( b_1, b_2, \ldots, b_k \) are linearly independent.

Note that this definition makes no explicit mention of dimension. However, saying that vectors \( b_1, b_2, \ldots, b_k \) are linearly independent is the same as saying that they span a \( k \)-dimensional subspace, and therefore \( S \) must have dimension \( k \).

The dimension of a subspace is equal to the number of vectors in any basis for the subspace.

If \( \{ b_1, b_2, \ldots, b_k \} \) is a basis for \( S \), then every vector \( v \) in \( S \) can be written as a linear combination of the basis vectors:

\[
    v = a_1 b_1 + a_2 b_2 + \cdots + a_k b_k.
\]

The coefficients \( a_1, a_2, \ldots, a_k \) are called the **components** of \( v \) with respect to the basis \( \{ b_1, b_2, \ldots, b_k \} \).

**EXAMPLE 1**

The vectors \( b_1 = (1, 1, 1) \) and \( b_2 = (7, 0, 2) \) form a basis for the plane \( 2x + 5y - 7z = 0 \). Find the components of the vector \( v = (-2, 5, 3) \) with respect to this basis.

**SOLUTION**

We wish to express \( v \) as a linear combination of \( b_1 \) and \( b_2 \), so we row reduce the matrix whose columns are \( b_1 \), \( b_2 \), and \( v \).

\[
\begin{bmatrix}
1 & 7 & -2 \\
1 & 0 & 5 \\
1 & 2 & 3 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & -2 \\
0 & -7 & 7 \\
0 & -5 & 5 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 5 \\
0 & 1 & -1 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

We conclude that \( v = 5b_1 - b_2 \), so the components are 5 and -1.

**Basis for a Solution Space**

As we have seen, the solution set to a homogeneous linear system in \( n \) variables is a subspace of \( \mathbb{R}^n \), which we refer to as the **solution space**. We already know how to use row reduction to find a set of vectors that span the solution space. It turns out that the vectors produced by this procedure are always linearly independent, and are therefore a basis for the solution space.
EXAMPLE 2

Find a basis for the solution space to the following linear system.

\[ \begin{align*}
  x_1 - x_2 + 2x_3 - 3x_4 &= 0 \\
  -3x_1 + 4x_2 - 2x_3 + 7x_4 &= 0
\end{align*} \]

SOLUTION

We row reduce the corresponding matrix.

\[
\begin{bmatrix}
  1 & -1 & 2 & -3 & 0 \\
  -3 & 4 & -2 & 7 & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
  1 & -1 & 2 & -3 & 0 \\
  0 & 1 & 4 & -2 & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
  1 & 0 & 6 & -5 & 0 \\
  0 & 1 & 4 & -2 & 0
\end{bmatrix}
\]

The variables \( x_3 \) and \( x_4 \) are free, and the solutions can be parameterized by

\[ \begin{align*}
  x_1 &= -6s + 5t, \\
  x_2 &= -4s + 2t, \\
  x_3 &= s, \\
  x_4 &= t.
\end{align*} \]

This can be written in vector form as follows.

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4
\end{bmatrix} = s \begin{bmatrix}
  -6 \\
  -4 \\
  1 \\
  0
\end{bmatrix} + t \begin{bmatrix}
  5 \\
  2 \\
  0 \\
  1
\end{bmatrix}
\]

Thus the vectors \((-6, -4, 1, 0)\) and \((-5, 2, 0, 1)\) span the solution space. Since these vectors are linearly independent, they are a basis for the solution space.

Thus the solution space is two-dimensional, i.e. a plane through the origin in \( \mathbb{R}^4 \).

The spanning vectors for a solution space that one obtains from the reduced echelon form are always a basis. For example, consider the following reduced echelon matrix.

\[
\begin{bmatrix}
  1 & 3 & 0 & -2 & 0 & -5 & 4 & 0 \\
  0 & 0 & 1 & 4 & 0 & 7 & -3 & 0 \\
  0 & 0 & 0 & 1 & -9 & 2 & 0 & 0
\end{bmatrix}
\]

The corresponding parametrization for the solution space is

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5 \\
  x_6 \\
  x_7
\end{bmatrix} = t_1 \begin{bmatrix}
  -3 \\
  1 \\
  0 \\
  0 \\
  0 \\
  0 \\
  0
\end{bmatrix} + t_2 \begin{bmatrix}
  2 \\
  0 \\
  -4 \\
  -7 \\
  0 \\
  0 \\
  0
\end{bmatrix} + t_3 \begin{bmatrix}
  5 \\
  0 \\
  0 \\
  0 \\
  9 \\
  0 \\
  1
\end{bmatrix} + t_4 \begin{bmatrix}
  -4 \\
  0 \\
  3 \\
  0 \\
  -2 \\
  0 \\
  1
\end{bmatrix}
\]

where \( t_1, t_2, t_3, \) and \( t_4 \) are parameters. It is easy to see that the four vectors on the right are linearly independent, since each vector has a 1 component in a position where the other three vectors have a 0 component. Thus these four vectors form a basis for the solution space.

Basis for a Span

If a subspace \( S \) is the span of a set of vectors \( \{v_1, v_2, \ldots, v_n\} \), it is always possible to find some subset of \( \{v_1, v_2, \ldots, v_n\} \) that is a basis for \( S \).

Specifically, we say that a vector \( v_i \) in the list \( v_1, v_2, \ldots, v_n \) is redundant if it can be expressed as a linear combination of the previous vectors. Then a basis for \( S \) can be obtained from \( \{v_1, v_2, \ldots, v_n\} \) by removing all of the redundant vectors.

The zero vector is always considered redundant, even it is the first vector in the list.
EXAMPLE 3

Find a basis for the subspace of $\mathbb{R}^4$ spanned by the following vectors.

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_5 = \begin{bmatrix} 8 \\ 4 \\ 0 \\ 0 \end{bmatrix}, \quad v_6 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

SOLUTION

Clearly $v_2$, $v_4$, and $v_5$ are redundant, since

$$v_2 = 2v_1, \quad v_4 = 5v_1 + 3v_3, \quad \text{and} \quad v_5 = 4v_1 + 8v_3.$$  

The remaining vectors $v_1$, $v_3$, and $v_6$ are clearly not redundant, and thus $\{v_1, v_3, v_6\}$ is a basis for this subspace.

It was obvious in the last example which vectors were redundant, but in general we can figure this out by row reducing the matrix whose columns are the given vectors. For example, if we put the six vectors $v_1, v_2, \ldots, v_6$ from the previous example into a matrix and row reduce, we get

$$\begin{bmatrix} 0 & 0 & 1 & 3 & 8 & 1 \\ 1 & 2 & 0 & 5 & 4 & 2 \\ 1 & 2 & 0 & 5 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} 1 & 2 & 0 & 5 & 4 & 0 \\ 0 & 0 & 1 & 3 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Here the second, fourth, and fifth columns have no pivots, which means that the original vectors $v_2$, $v_4$, and $v_5$ were redundant. The first, third, and sixth columns do have pivots, which means that $\{v_1, v_3, v_6\}$ are a basis for the subspace.

EXAMPLE 4

Find a basis for the subspace of $\mathbb{R}^4$ spanned by the following vectors.

$$\begin{bmatrix} 1 \\ 3 \\ 3 \\ 3 \\ 4 \\ 4 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \\ 3 \\ 1 \\ 8 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 1 \\ 0 \\ -1 \\ 11 \\ 14 \\ 5 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ -5 \\ -2 \\ -5 \end{bmatrix}$$

SOLUTION

We put the vectors into the columns of a matrix and row reduce.

$$\begin{bmatrix} 1 & 3 & 3 & 4 & 3 \\ 1 & 2 & 1 & 1 & 2 \\ 2 & 3 & 0 & -1 & 4 \\ 2 & 7 & 8 & 11 & 14 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 3 & 4 & 4 \\ 1 & 3 & -2 & -3 & -2 \\ 0 & -3 & -6 & 0 & -9 \\ 0 & 1 & 2 & 3 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -3 & -5 & -2 \\ 0 & 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

Row reduction does not change the relationship between the columns of a matrix.

It follows that the given subspace of $\mathbb{R}^4$ is three-dimensional.
The first, second, and fifth columns have pivots, and therefore the original first, second and fifth vectors are a basis for the subspace. Thus a basis is

\[
\{(1, 1, 2), (3, 2, 3, 7), (4, 2, 4, 14)\}
\]